13.1 Curves in Space and Tangents

A real valued function $f$:

$\begin{align*}
\text{Domain} &= \mathbb{R} \\
\text{Range} &= \mathbb{R}
\end{align*}$

A vector valued function $\mathbf{F}$:

$\begin{align*}
\text{Domain} &= \mathbb{R} \\
\text{Range} &= \text{vectors}
\end{align*}$

$\mathbf{F}(t) = \langle f(t), g(t), h(t) \rangle$

Curves in space are described through vector valued functions.

Example. A line is a curve in space, and recall that we describe lines through parametric equations

$\begin{align*}
\mathbf{r}(t) &= (x_0 + tv_1, y_0 + tv_2, z_0 + tv_3) \\
\mathbf{F}(t) &= \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle
\end{align*}$

(line through $(x_0, y_0, z_0)$ parallel to $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$)

In general if $x=f(t)$, $y=g(t)$ and $z=h(t)$ for some functions $f$, $g$ and $h$, we have a set of points in space obtained by varying the parameter $t$.

* If all three of $f$, $g$ and $h$ are continuous functions, then the curve they describe is continuous.

E.g. $\mathbf{F}(t) = (4 + \sin(20t)) \cos(t) \mathbf{i} + (4 + \sin(20t)) \sin(t) \mathbf{j} + \cos(20t) \mathbf{k}$
Typically we use $\vec{r}(t)$, with $t$ in some interval $I$ to denote vector equations of curves.

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Point $(x_o, y_o, z_o)$, corresponding to some $t_o$ its position vector is: $\vec{r}(t_o) = \langle x_o, y_o, z_o \rangle$

**Example. The Helix**

Sketch the vector function (curve) given by: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$

Note that $x(t) = \cos t$ and $y(t) = \sin t$ satisfy:

$$\cos^2 t + \sin^2 t = \frac{x^2 + y^2}{x^2 + y^2} = 1$$

i.e. all points of the curve lie on the cylinder $x^2 + y^2 = 1$

If $0 \leq t \leq 2\pi$

- $\vec{r}(0) = \langle -1, 0, 0 \rangle$
- $\vec{r}(\pi/2) = \langle 0, 1, \pi/2 \rangle$
- $\vec{r}(\pi) = \langle 1, 0, \pi \rangle$
- $\vec{r}(3\pi/2) = \langle -1, 0, \pi \rangle$

If $-\infty < t < \infty$
Limits and Continuity

To work with vector valued functions \( \mathbf{r}(t) = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k} \), we do it component by component.

Limits: We define \( \lim_{t \to t_0} \mathbf{r}(t) = \left( \lim_{t \to t_0} f(t), \lim_{t \to t_0} g(t), \lim_{t \to t_0} h(t) \right) \)

Definition. Consequently, we say \( \mathbf{r}(t) \) is continuous at \( t_0 \) if \( \lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0) \)

And \( \mathbf{r}(t) \) is continuous if it is continuous at every \( t \) in its domain, which is in fact: \( \text{Dom } \mathbf{r} = \text{Dom } f \cap \text{Dom } g \cap \text{Dom } h \).

Derivative of \( \mathbf{r}(t) \)

\[ \mathbf{PQ} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) =: \Delta \mathbf{r} \]

\[ \frac{\Delta \mathbf{r}}{\Delta t} \] is a scalar multiple of \( \Delta \mathbf{r} \)

Note as \( \Delta t \to 0 \), \( \frac{\Delta \mathbf{r}}{\Delta t} \to \text{tangent vector at } P \)

That is \( \mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \hat{i} + \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \hat{j} + \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} \hat{k} = f'(t) \hat{i} + g'(t) \hat{j} + h'(t) \hat{k} \)

That is: \( \mathbf{r}'(t) = \left< f'(t), g'(t), h'(t) \right> \) as long as \( f'(t), g'(t) \) and \( h'(t) \) exist.
Motion in Space

\( r'(t) \) (wherever it is non zero) is the vector tangent to the curve \( \vec{r}(t) \) at \( t \) and it points in the direction of motion.

In particular this tells us that the same curve can be parametrized in distinct directions.

Consider a particle traveling with a path described by the curve:

\[
\vec{r}(t) = \langle f(t), g(t), h(t) \rangle
\]

meaning it's position at time \( t \) is given by \( \vec{r}(t) \), then:

1. The velocity of the particle is:
   \[
   \vec{v}(t) = \vec{r}'(t)
   \]

2. Its speed is given by:
   \[
   |\vec{v}(t)| = |\vec{r}'(t)|
   \]

3. Its acceleration is given by:
   \[
   \vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)
   \]

4. The unit vector \( \frac{\vec{v}}{|\vec{v}|} \) is the direction of motion at time \( t \).

Example. Find the speed and acceleration of a particle whose motion is given by the position vector: