Example. Find parametric equations for the line through the points $P(-3,2,-3)$ and $Q(1,-1,4)$

- Direction vector: $\vec{PQ} = \langle 4, -3, 7 \rangle = \vec{v}$
- Point: $P_0 = P(-3, 2, -3)$

Parametric Equations:

\[
\begin{align*}
    x &= -3 + 4t \\
    y &= 2 - 3t \\
    z &= 3 + 7t
\end{align*}
\]

Example. Find the line through $P(2,1,0)$ and orthogonal to both $\vec{u} = \hat{i} + \hat{j} = \langle 1, 1, 0 \rangle$ and $\vec{w} = \hat{j} + \hat{k} = \langle 0, 1, 1 \rangle$

Direction vector $\vec{v} = \vec{u} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \hat{i} - \hat{j} + \hat{k}$

Parametric Equations:

\[
\begin{align*}
    x &= 2 + t \\
    y &= 1 - t \\
    z &= 0 + t
\end{align*}
\]
Example  A helicopter is to fly directly from a helipad at the origin in the direction of the point (1,1,1) at the speed of 60 ft/s. What is its position after 10 seconds?

\[ \text{direction & length.} \]
\[ \text{direction } \vec{s} = \frac{\langle 1, 1, 1 \rangle}{||\langle 1, 1, 1 \rangle||} = \frac{\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle}{\text{unit vector}} \]

\[ \text{distance traveled in } t \text{ seconds } = 60t \]
\[ \text{Position after } t \text{ seconds } = 60t \langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle \]
\[ \text{after 10 seconds } \text{ position } = \langle \frac{600}{\sqrt{3}}, \frac{600}{\sqrt{3}}, \frac{600}{\sqrt{3}} \rangle \]
\[ \text{or } = \langle 200\sqrt{3}, 200\sqrt{3}, 200\sqrt{3} \rangle \]

Distance from a point to a line  Consider the line through \( P \) and parallel to \( \vec{v} \).

The distance we are looking for is the distance from the point S to Q.

\[ d(S, L) = ||\vec{PS}|| \sin \theta = \frac{||\vec{PS} \times \vec{v}||}{||\vec{v}||} \]
Equation of a plane through a point \( P_0 \) and with normal \( \vec{n} \)

For a point \( P(x,y,z) \) to lie on the plane it is necessary and sufficient that \( \vec{P_0P} \) is perpendicular to \( \vec{n} \).

Thus the equation for a plane through \( P \) and with normal vector \( \vec{n} = A\hat{i} + B\hat{j} + C\hat{k} \) is:

\[
\vec{P_0P} = \langle x-x_0, y-y_0, z-z_0 \rangle
\]

\[
\vec{P_0P} \cdot \vec{n} = 0 \quad \text{(vector equation)}
\]

\[
\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \langle A, B, C \rangle = 0
\]

\[
A(x-x_0) + B(y-y_0) + C(z-z_0) = 0
\]

\[
Ax + By + Cz + D = 0
\]

**Example** Find the equation of a plane through the points \( P(0,0,1), Q(2,0,0) \) and \( R(0,3,0) \)

\[
\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = \hat{i}(3) - \hat{j}(-2) + \hat{k}(6)
\]

\[
\hat{i} = \langle 2, 0, -1 \rangle \quad \hat{j} = \langle 0, 3, -1 \rangle
\]

Use \( P \) as the point on plane. \( \langle x_0, y_0, z_0 \rangle = \langle 0, 0, 1 \rangle \)

\[
3(x-0) + 2(y-0) + 6(z-1) = 0
\]

\[
3x + 2y + 6z - 6 = 0
\]
Example: Find the parametric equations of the line of intersection of the planes: \( \pi_1: 3x - 6y - 2z = 15 \) and \( \pi_2: 2x + y - 5z = 5 \)

\[
\vec{n}_1 = \langle 3, -6, -2 \rangle \quad \quad \vec{n}_2 = \langle 2, 1, -5 \rangle
\]

First: find a direction vector for the line of intersection:

\[
\overrightarrow{u} \text{ lies on } \pi_1 \Rightarrow \overrightarrow{u} \perp \vec{n}_1 \quad \text{Thus use } \overrightarrow{u} = \vec{n}_1 \times \vec{n}_2
\]

\[
\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
3 & -6 & -2 \\
2 & 1 & -5
\end{vmatrix} = \hat{i}(32) - \hat{j}(-11) + \hat{k}(15) = \overrightarrow{u}
\]

or \( \overrightarrow{u} = \langle 32, 11, 15 \rangle \)

Second: Find a point on the line. Let \( z = 0 \) (we'll find the point of intersection of the line w/ plane \( xy \)).

Substitute into eqns of planes:

\[
\begin{align*}
3x - 6y &= 15 \\
2x + y &= 5
\end{align*}
\]

Solve system

\[
\begin{align*}
x &= 3 \\
y &= -1
\end{align*}
\]

Thus, \( (3, -1, 0) \) is on both planes, hence on the line of intersect.

The parametric equations are:

\[
\begin{align*}
x &= 3 + 32t \\
y &= -1 + 11t \\
z &= 0 + 15t
\end{align*}
\]

We can also find the angle between the planes:

Recall \( \theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{||\vec{n}_1|| \cdot ||\vec{n}_2||}\right) \)

The angle between the normal vectors is also the angle between the planes.
Example  Find the point where the line given by:  \( x = \frac{2}{3} + 2t \),  
\( y = -2t \),  \( z = 1 + t \)

intersects the plane  \( 3x + 2y + 6z = 6 \)

\[ 3\left(\frac{2}{3} + 2t\right) + 2\left(-2t\right) + 6\left(1 + t\right) = 6 \]
\[ 8t + 6t - 4t + 6 + 6t = 6 \]
\[ 8t = 6 - 6 - 8 \]
\[ 8t = -8 \rightarrow t = -1 \]

\( x(-1) = \frac{2}{3} - 2 = \frac{2}{3} \)
\( y(-1) = -2(-1) = 2 \)
\( z(-1) = 1 - 1 = 0 \)

Plug in \( x(t) \), \( y(t) \) and \( z(t) \) into the equation of the plane and solve for \( t \).

Distance from a point to a plane

Consider the plane through \( P \) and with normal \( \vec{n} \), the distance we need is the length of the projection of \( PS \) onto the line through \( P \) and parallel to \( n \)

\[
\quad \quad \quad \quad \quad \quad \quad d(S, \text{plane}) = \left| \frac{\vec{PS} \cdot \vec{n} \cos \theta}{|\vec{n}|} \right|
\]

End of 12.5
A cylinder:

In general, a cylinder is any surface that does not change as we move in some fixed direction.

**Definition** A cylinder is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a generating curve for the cylinder.

Equation for this cylinder:

\[ z = y^2 \]

(x · missing from the equation)
Example

Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola: \( y = x^2, \ z = 0 \)

In general

- Cylinder parallel to 2-axis with generating curve \( f(x, y) = c \) (on xy-plane)

Similarly:
- Cylinder parallel to x-axis with generating curve \( f(y, z) = c \) (on yz-plane)
- Cylinder parallel to y-axis with generating curve \( f(x, z) = c \) (on xz-plane)

Thus for instance, the equation:
(in 3-space)

\[ x^2 + z^2 = 4 \]
**Quadratic Surfaces**

A quadratic surface is the graph in space of a second-degree equation in $x$, $y$ and $z$.

To understand/visualize these surfaces we can consider their cross sections, that is, the intersection curves of the surface with planes parallel to the coordinate planes.

Let $k$ be a constant real number:

- A plane parallel to $xy$-plane has equation: $z = k$
- A plane parallel to $xz$-plane has equation: $y = k$
- A plane parallel to $yz$-plane has equation: $x = k$

**Example**  
**The Ellipsoid**

Equation: \[
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1
\]

Intersection with the coordinate planes:

- $z = 0$ ($xy$-plane)
- $y = 0$
- $x = c$ ($xz$-plane)
- $\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ ($yz$-plane)
The Hyperbolic Paraboloid $Z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$

<table>
<thead>
<tr>
<th>Cross Sections</th>
<th>Parallel to</th>
<th>fix</th>
<th>Equation</th>
<th>curve</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$yz$-plane</td>
<td>$x$</td>
<td>$Z = \frac{y^2}{b^2} - K$</td>
<td>parabolas</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$xz$-plane</td>
<td>$y$</td>
<td>$Z = \frac{-x^2}{a^2} + K$</td>
<td>parabolas</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$xy$-plane</td>
<td>$z$</td>
<td>$Z = \frac{y^2}{kb^2} - \frac{x^2}{ka^2} = 1$</td>
<td>Hyperbolas</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[\begin{align*}
\text{Graph} & : \quad \frac{x}{a} > \frac{y}{b} > 1 \\
\text{OK} & : \quad \frac{x}{a} < \frac{y}{b} < 1
\end{align*}\]
The Elliptic Paraboloid

\[ z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \]

<table>
<thead>
<tr>
<th>Cross sections parallel to</th>
<th>( f_x )</th>
<th>Equation</th>
<th>Curve</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( yz )-plane</td>
<td>( x )</td>
<td>[ z = \frac{y^2}{b^2} + k ]</td>
<td>parabolas</td>
<td><img src="yz-plane.png" alt="Graph" /></td>
</tr>
<tr>
<td>( xz )-plane</td>
<td>( y )</td>
<td>[ z = \frac{x^2}{a^2} + k ]</td>
<td>parabolas</td>
<td><img src="xz-plane.png" alt="Graph" /></td>
</tr>
<tr>
<td>( xy )-plane</td>
<td>( z )</td>
<td>[ \frac{x^2}{k_a^2} + \frac{y^2}{k_b^2} = 1 ]</td>
<td>Ellipse</td>
<td><img src="xy-plane.png" alt="Graph" /></td>
</tr>
</tbody>
</table>

On \( xy \)-plane \( z = 0 \), \( x^2 + y^2 = 0 \)
ex. The cone \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} \)

<table>
<thead>
<tr>
<th>Cross sections (//) to</th>
<th>Fix</th>
<th>Equation</th>
<th>Curve</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>yz-plane</td>
<td>x</td>
<td>( \frac{z^2}{c^2} - \frac{y^2}{b^2} = k )</td>
<td>Hyperbolas</td>
<td><img src="hyperbola.png" alt="Graph" /></td>
</tr>
<tr>
<td>xz-plane</td>
<td>y</td>
<td>( \frac{z^2}{c^2} - \frac{x^2}{a^2} = k )</td>
<td>Hyperbolas</td>
<td><img src="hyperbola.png" alt="Graph" /></td>
</tr>
<tr>
<td>xy-plane</td>
<td>z</td>
<td>( \frac{x^2}{a^2} + \frac{y^2}{b^2} = k )</td>
<td>Ellipses or a pt if (z=0)</td>
<td><img src="ellipsoid.png" alt="Graph" /></td>
</tr>
</tbody>
</table>
The Hyperboloid of one sheet \( \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \)

<table>
<thead>
<tr>
<th>Cross Sections ( \parallel ) to</th>
<th>fix</th>
<th>Equation</th>
<th>curve</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>( yz )-plane</td>
<td>( x )</td>
<td>( \frac{y^2}{b^2} - \frac{z^2}{c^2} = k )</td>
<td>hyperbolas</td>
<td>( y )</td>
</tr>
<tr>
<td>( xz )-plane</td>
<td>( y )</td>
<td>( \frac{x^2}{a^2} - \frac{z^2}{c^2} = k )</td>
<td>hyperbolas</td>
<td>( x )</td>
</tr>
<tr>
<td>( xy )-plane</td>
<td>( z )</td>
<td>( \frac{x^2}{a^2} + \frac{y^2}{b^2} = k )</td>
<td>Ellipses</td>
<td>( x )</td>
</tr>
</tbody>
</table>
The hyperboloid of 2 sheets \[ \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \]

<table>
<thead>
<tr>
<th>Cross sections \parallel to</th>
<th>fix</th>
<th>Equation</th>
<th>curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>yz-plane</td>
<td>x</td>
<td>( \frac{y^2}{b^2} + \frac{z^2}{c^2} = k )</td>
<td>Ellipses</td>
</tr>
<tr>
<td>xz-plane</td>
<td>y</td>
<td>( \frac{x^2}{a^2} - \frac{z^2}{c^2} = k )</td>
<td>Hyperbolas</td>
</tr>
<tr>
<td>xy-plane</td>
<td>z</td>
<td>( \frac{x^2}{a^2} - \frac{y^2}{b^2} = k )</td>
<td>Hyperbolas</td>
</tr>
</tbody>
</table>