Homework #01, due 1/20/10 = 9.1.2, 9.1.4, 9.1.6, 9.1.8, 9.2.3

Additional problems for study: 9.1.1, 9.1.3, 9.1.5, 9.1.13, 9.2.1, 9.2.2, 9.2.4, 9.2.5, 9.2.6, 9.3.2, 9.3.3

9.1.1 (This problem was not assigned except for study, but it’s useful for the next problem.) Let $p$ and $q$ be polynomials in $\mathbb{Z}[x, y, z]$, where

\[
p = p(x, y, z) = 2x^2y - 3xy^3z + 4y^2z^5
\]
\[
q = q(x, y, z) = 7x^2 + 5x^2y^3z^4 - 3x^2z^3
\]

(a) Write each of $p$ and $q$ as a polynomial in $x$ with coefficients in $\mathbb{Z}[y, z]$.

\[
p = (2y)x^2 - (3y^3z)x + 4y^2z^5
\]
\[
q = (7 + 5y^3z^4 - 3z^3)x^2
\]

(b) Find the degree of each of $p$ and $q$.

\[
\deg(p) = 7 \quad \deg(q) = 9
\]

(c) Find the degree of $p$ and $q$ in each of the three variables $x, y, \text{ and } z$.

\[
\deg_x(p) = 2 \quad \deg_x(q) = 2
\]
\[
\deg_y(p) = 3 \quad \deg_y(q) = 3
\]
\[
\deg_z(p) = 5 \quad \deg_z(q) = 4
\]
(d) Compute \( pq \) and find the degree of \( pq \) in each of the three variables \( x, y, \) and \( z \).

\[
pq = \left(2x^2y - 3xy^3z + 4y^2z^5\right)\left(7x^2 + 5x^2y^3z^4 - 3x^2z^3\right)
\]
\[
= 2x^2y\left(7x^2 + 5x^2y^3z^4 - 3x^2z^3\right)
- 3xy^3z\left(7x^2 + 5x^2y^3z^4 - 3x^2z^3\right)
+ 4y^2z^5\left(7x^2 + 5x^2y^3z^4 - 3x^2z^3\right)
\]
\[
= 2x^2y7x^2 + 2x^2y5x^2y^3z^4 - 2x^2y3x^2z^3
- 3xy^3z7x^2 - 3xy^3z5x^2y^3z^4 + 3xy^3z3x^2z^3
+ 4y^2z^57x^2 - 4y^2z^55x^2y^3z^4 - 4y^2z^53x^2z^3
\]
\[
= 14x^4y + 10x^4y^4z^4 - 6x^4yz^3
- 21x^3y^3z - 15x^3y^6z^5 + 9x^3y^4z^4
+ 28x^2y^2z^5 + 20x^2y^5z^9 - 12x^2y^2z^8
\]

\[
\deg_x(pq) = 2 + 2 = 4 \quad \deg_y(pq) = 3 + 3 = 6 \quad \deg_z(pq) = 5 + 4 = 9
\]

(e) Write \( pq \) as a polynomial in the variable \( z \) with coefficients in \( \mathbb{Z}[x, y] \).

\[
pq = 14x^4y - (21x^3y^3)z - (6x^4y)z^3 + (10x^4y^4 + 9x^3y^3)z^4
+ (28x^2y^2 - 15x^3y^6)z^5 - (12x^2y^2)z^8 + (20x^2y^5)z^9
\]

9.1.2 Repeat the preceding exercise under the assumption that the coefficients of \( p \) and \( q \) are in \( \mathbb{Z}/3\mathbb{Z} \). Let \( p \) and \( q \) be polynomials in \( \mathbb{Z}/3\mathbb{Z}[x, y, z] \), where

\[
p = p(x, y, z) = \overline{2}x^2y + y^2z^5
q = q(x, y, z) = x^2 + \overline{2}x^2y^3z^4
\]

(a) Write each of \( p \) and \( q \) as a polynomial in \( x \) with coefficients in \( \mathbb{Z}/3\mathbb{Z}[y, z] \).
We need only reduce the coefficients modulo 3, which gives us
\[ p = (\overline{2}y)x^2 + y^2z^5 \]
\[ q = (\overline{1} + \overline{2}y^3z^4)x^2 \]

(b) Find the degree of each of \( p \) and \( q \).
\[ \text{deg}(p) = 7 \quad \text{deg}(q) = 9 \]

(c) Find the degree of \( p \) and \( q \) in each of the three variables \( x \), \( y \), and \( z \).
\[ \text{deg}_x(p) = 2 \quad \text{deg}_x(q) = 2 \]
\[ \text{deg}_y(p) = 2 \quad \text{deg}_y(q) = 3 \]
\[ \text{deg}_z(p) = 5 \quad \text{deg}_z(q) = 4 \]

(d) Compute \( pq \) and find the degree of \( pq \) in each of the three variables \( x \), \( y \), and \( z \).
\[ pq = (\overline{2}x^2y + y^2z^5)(x^2 + 2x^2y^3z^4) \]
\[ = \overline{2}x^2y(x^2 + 2x^2y^3z^4) + y^2z^5(x^2 + 2x^2y^3z^4) \]
\[ = 2x^2y(x^2) + 2x^2y(2x^2y^3z^4) + y^2z^5x^2 + y^2z^5(2x^2y^3z^4) \]
\[ = 2x^4y + x^4y^4z^4 + x^2y^2z^5 + 2x^2y^5z^9 \]
\[ \text{deg}_x(pq) = 2 + 2 = 4 \quad \text{deg}_y(pq) = 3 + 3 = 6 \quad \text{deg}_z(pq) = 5 + 4 = 9 \]

(e) Write \( pq \) as a polynomial in the variable \( z \) with coefficients in \( \mathbb{Z}/3\mathbb{Z}[x, y] \).
\[ pq = \overline{2}x^4y + x^4y^4z^4 + x^2y^2z^5 + 2x^2y^5z^9 \]

9.1.4 Prove that the ideals \((x)\) and \((x, y)\) are prime ideals in \( \mathbb{Q}[x, y] \) but only the latter ideal is a maximal ideal.

To show \((x)\) is a prime ideal of \( \mathbb{Q}[x, y] \), we must assume \( pq \in (x) \), where \( p, q \in \mathbb{Q}[x, y] \), and show that either \( p \in (x) \) or \( q \in (x) \). Suppose \( p = a_nx^n + \cdots + a_1x + a_0 \) and \( p = b_nx^m + \cdots + b_1x + b_0 \), where \( a_n, \ldots, a_0, b_m, \ldots, b_0 \in \mathbb{Q}[y] \). Then \( pq \) has exactly one term with \( x \)-degree 0, namely the polynomial \( a_0b_0 \in \mathbb{Q}[y] \).
From the assumption $pq \in (x)$ we know there is some polynomial $s \in \mathbb{Q}[x, y]$ such that $pq = sx$. Note that every (non-zero) term in $sx$ has $x$ as a factor, hence its $x$-degree is at least 1. Hence every nonzero term in $pq$ has degree at least 1. It therefore follows from $pq = sx$ that $a_0b_0 = 0$.

Now $\mathbb{Q}$ is an integral domain, hence $\mathbb{Q}[y]$ is also an integral domain by Prop. 9.2, so from $a_0b_0 = 0$ we conclude that either $a_0 = 0$ or $b_0 = 0$. If $a_0 = 0$ then every term of $p$ has $x$ as a factor, hence $p$ is equal to $x$ multiplied by some polynomial, so $p \in (x)$. Similarly, if $b_0 = 0$, then $q \in (x)$. This shows that $p$ or $q$ is in $(x)$, so $(x)$ is prime.

To show $(x, y)$ is a prime ideal of $\mathbb{Q}[x, y]$, we must assume $pq \in (x, y)$, where $p, q \in \mathbb{Q}[x, y]$, and show that either $p \in (x, y)$ or $q \in (x, y)$. From $pq \in (x, y)$ we get $pq = sx + ty$ for some polynomials $s, t \in \mathbb{Q}[x, y]$. This means that the constant term of $pq$ is 0, because $sx + ty$ is a polynomial in which every term either has $x$ as a factor or has $y$ as a factor. Let $a_0 \in \mathbb{Q}$ be the constant term of $p$ and let $b_0 \in \mathbb{Q}$ be the constant term of $q$. Then $a_0b_0 = 0$, so either $a_0 = 0$ or $b_0 = 0$. But if $a_0 = 0$ then every term of $p$ has either $x$ or $y$ (or both) as a factor, and this implies that $p \in (x, y)$. Similarly, if $b_0 = 0$ then $q \in (x, y)$. This shows that $(x, y)$ is prime.

Define a function $\varphi : \mathbb{Q} \to \mathbb{Q}[x, y]/(x, y)$ by $\varphi(q) = q + (x, y)$ for every $q \in \mathbb{Q}$. It is easy to show that $\varphi$ preserves $+$ and $\cdot$ and is therefore a ring homomorphism. Note that $\varphi$ sends each rational $q$ to the set of polynomials in $\mathbb{Q}[x, y]$ which have constant term equal to $q$. Now every polynomial in $\mathbb{Q}[x, y]$ has a constant term, so $\varphi$ is onto. Furthermore, polynomials with distinct constant terms are distinct, so $\varphi$ is injective. These considerations show that $\mathbb{Q}[x, y]/(x, y)$ is isomorphic to $\mathbb{Q}$. But $\mathbb{Q}$ is a field, so the ideal $(x, y)$ is a maximal ideal by Prop. 7.12 (which says, if $R$ a commutative ring then an ideal $M$ is maximal iff $R/M$ is a field).

One the other hand, $(x)$ is not a maximal ideal because (as we will show) it is a proper subset of the maximal ideal $(x, y)$. We have $(x) \subseteq (x, y)$ since $\{x\} \subseteq \{x, y\}$. To show the inclusion is proper it is enough
to note that \( y \in (x, y) \) but \( y \notin (x) \) because the \( y \)-degree of \( y \) is 1 but the \( y \)-degree of every polynomial in \( (x) \) is 0.

9.1.6 Prove that \((x, y)\) is not a principal ideal in \( \mathbb{Q}[x, y] \).

Let us suppose, to the contrary, that \((x, y) = (p)\) for some polynomial \( p \in \mathbb{Q}[x, y] \). Then, since \( x, y \in (x, y) = (p) \), there are \( s, t \in \mathbb{Q}[x, y] \) such that \( x = sp \) and \( y = tp \). Then \( 0 = \deg_y(x) = \deg_y(s) + \deg_y(p) \), so \( 0 = \deg_y(p) \), and \( 0 = \deg_x(y) = \deg_x(t) + \deg_x(p) \), so \( 0 = \deg_x(p) \). From \( 0 = \deg_y(p) = \deg_x(p) \) we get \( \deg(p) = 0 \) and \( p \in \mathbb{Q} \). But \( p \in (p) = (x, y) \), so \( p = fx + gy \) for some \( f, g \in \mathbb{Q}[x, y] \), so \( \deg(p) = \deg(fx + gy) = \min(\deg(f) + \deg(x), \deg(g) + \deg(y)) = \min(\deg(f) + 1, \deg(g) + 1) \geq 1 \), contradicting \( \deg(p) = 0 \).

9.1.8 Let \( F \) be a field and let \( R = F[x, x^2y, x^3y^2, \ldots, x^n y^{n-1}, \ldots] \) be a subring of the polynomial ring \( F[x, y] \).

(a) Prove that the field of fractions of \( R \) and \( F[x, y] \) are the same.

Just as a review, we note that Th. 7.15 says, if \( R \) is a commutative ring, \( \emptyset \neq D \subseteq R, 0 \notin D, D \) is closed under \( \cdot \), and \( D \) contains no 0-divisors, then there is a ring of quotients \( Q = D^{-1}R \supseteq R \) such that

1. \( Q \) is a commutative ring with 1,
2. \( D \subseteq Q^\times \),
3. \( Q = \{d^{-1}r | d \in D, r \in R\} \),
4. if \( D \) is a set of units in \( S \supseteq R \), a commutative ring with 1 extending \( R \), then \( Q \) is isomorphic to a subring \( Q' \) of \( S \) extending \( R \), i.e., \( Q \cong Q' \) and \( R \subseteq Q' \subseteq S \).

If \( R \) is an integral domain, then \( D = R - \{0\} \) has the needed properties and \( D^{-1}R \) is a field, called the field of quotients of \( R \).

If \( F \) is a field, then the polynomial ring \( F[x, y] \) is an integral domain, so \( F[x, y] \) has a field of fractions obtained by choosing \( D = F[x, y] - \{0\} \). Let \( Q_1 \) be the field of fractions of \( F[x, y] \). Then

\[
R = F[x, x^2y, x^3y^2, \ldots, x^n y^{n-1}, \ldots] \subseteq F[x, y] \subseteq Q_1 = D^{-1}F[x, y]
\]
By Theorem 7.15, there is a subring $Q_2$ of $Q_1$ which is isomorphic to the field of quotients of $R$, so

$$R = F[x, x^2y, x^3y^2, \ldots, x^n y^{n-1}, \ldots] \subseteq Q_2 \subseteq Q_1$$

and we need only show $Q_1 \subseteq Q_2$.

Note that $x \in R$ so $x \in Q_2$. Also, $0 \neq x, x^2y \in R$ so $x, x^2y \in Q_2$, but $x$ is a unit of $Q_2$, so $y = (x^{-1})^2 x^2 y \in Q'$. From $x, y \in Q_2$ we get

$$F[x, y] \subseteq Q_2.$$

Consider an arbitrary element $d^{-1}p \in Q_1$, where $0 \neq d \in F[x, y]$ and $p \in F[x, y]$. Then $d, p \in Q_2$ and $d \neq 0$ and $Q_2$ is a field, so $d^{-1} = \frac{b}{a} \in Q_2$. This shows $Q_1 \subseteq Q_2$, completing the proof that $Q_1 = Q_2$.

(b) Prove that $R$ contains an ideal that is not finitely generated.

For every $n \in \mathbb{Z}^+$ let $G_n := \{x, \cdots, x^n y^{n-1}\}$, and let $\langle G_n \rangle$ be the closure of $G_n$ under multiplication. Then a polynomial $f \in F[x, y]$ belongs to $F[x, x^2y, x^3y^2, \ldots, x^n y^{n-1}]$ if and only if $f$ is a linear combination of a finite subset of $\langle G_n \rangle$, that is, there is a finite set of monomials $M \subseteq \langle G_n \rangle$ and a finite set of coefficients $\alpha_m \in F$, $m \in M$, such that $f = \sum_{m \in M} \alpha_m m$.

For every $f \in F[x, y]$ let $r(f) = \deg_x(f) - \deg_y(f)$. Note that if $m \in G_n$ then $r(m) = 1$. If $f, g \in F[x, y]$ then

$$r(fg) = \deg_x(fg) - \deg_y(fg)$$

$$= \deg_x(f) + \deg_x(g) - (\deg_y(f) + \deg_y(g))$$

$$= \deg_x(f) - \deg_y(f) + \deg_x(g) - \deg_y(g)$$

$$= r(f) + r(g)$$

Therefore, for every $m \in \langle G_n \rangle$, either $m \in G_n$ and $r(m) = 1$, or else $r(m) > 1$ (because $m$ is a nontrivial product of two or more monomials from $G_n$).

Claim $x^{n+1}y^n \notin F[x, \cdots, x^n y^{n-1}]$.

Proof Suppose, to the contrary, that $x^{n+1}y^n \in F[x, \cdots, x^n y^{n-1}]$. Then there is a finite set of monomials $M \subseteq \langle G_n \rangle$ and a finite set of
coefficients $\alpha_m \in F$, $m \in M$, such that $x^{n+1}y^n = \sum_{m \in M} \alpha_m m$. Now two polynomials in $F[x, y]$ are equal iff they have the same coefficients. In this case, this means that $x^{n+1}y^n$ is one of the monomials in $M$, say $x^{n+1}y^n = m \in M$, and its coefficient is $\alpha_m = 1$, and all the other coefficients are 0. Now $m \in M \subseteq \langle G_n \rangle$, but $r(m) = r(x^{n+1}y^n) = 1$, so in fact $m \in G_n$. However, this is a contradiction because $x^{n+1}y^n \notin G_n$.

$R$ is an ideal of itself, so by the following claim $R$ contains an ideal (itself) that is not finitely generated.

**Claim** $R$ is not finitely generated.

**Proof** Suppose, to the contrary, that $G \subseteq R$ is a finite set of generators of $R$. Now $R$ is the union of a chain of subrings since

$$R = F[x, x^2y, \ldots, x^ny^{n-1}, \ldots] = \bigcup_{n \in \mathbb{Z}^+} F[x, \ldots, x^ny^{n-1}].$$

Each element of $G$ is in one of the subrings $F[x, \ldots, x^ny^{n-1}] \subseteq R$, $n = 1, 2, 3, \ldots$. For each element of $G$, choose such a subring containing that element. There are only finitely many such subrings since $G$ is finite, and one of them, say $F[x, \ldots, x^N y^{N-1}]$ for some $N \in \mathbb{Z}^+$, contains all the others (because every finite subset of a linearly ordered set has a maximum element). Thus $G \subseteq F[x, \ldots, x^N y^{N-1}]$, so, since $G$ generates $R$, we have $R \subseteq F[x, \ldots, x^N y^{N-1}]$, but $F[x, \ldots, x^N y^{N-1}] \subseteq R$, so $F[x, \ldots, x^N y^{N-1}] = R$. We now have a contradiction since, by the claim above, $x^{N+1}y^N \notin F[x, \ldots, x^N y^{N-1}]$.

9.2.3 Let $f(x)$ be a polynomial in $F[x]$. Prove that $F[x]/(f(x))$ is a field if and only if $f(x)$ is irreducible. [Use Proposition 7, Section 8.2]

We note that Prop. 8.7 says, “Every nonzero prime ideal in a PID is maximal”. The proof actually shows that, in an integral domain, prime ideals are maximal among principal ideals. In a PID, all ideals are principal, so prime ideals are maximal.

We assume (although it is not explicitly mentioned in the text of the problem) that $F$ is a field. Then $F[x]$ is a Euclidean domain, hence $F[x]$ is a PID, so by Prop. 8.7, prime ideals in $F[x]$ are maximal.
Prop. 7.14 says that, in a commutative ring, every maximal ideal is prime. Therefore,

(1) in $F[x]$ an ideal is prime iff it is maximal.

Now complete the proof as follows—

$F[x]/(f(x))$ is a field

$\iff (f(x))$ is a maximal ideal of $F[x]$ \hspace{1cm}$\iff (f(x))$ is a prime ideal of $F[x]$ \hspace{1cm}$\iff f(x)$ is prime in $F[x]$ \hspace{1cm}$\iff f(x)$ is irreducible in $F[x]$  

Prop. 7.12 says, “If $R$ a commutative ring then an ideal $M$ is maximal iff $R/M$ is a field.” Prop. 8.11 says, “In a PID a nonzero element is a prime iff it is irreducible.”