Thus every automorphism of $K$ characteristic in $G$ carries the normal subgroup $1 \times G$ to itself. Let $\sigma$ be an automorphism of $G$ that interchanges the two factors, that is, $\sigma(g) = g'$ for all $g, g' \in G$. This automorphism carries the normal subgroup $1 \times G$ to itself, hence it is left unchanged by this automorphism, hence $\sigma(g) = g$, as desired.

(b) Prove that if $H$ is characteristic in $K$ and $K$ is characteristic in $G$ then $H$ is characteristic in $G$. Use this to prove the Klein 4-group $V_4$ is characteristic in $S_4$.

Let $\sigma$ be an automorphism of $G$. Then $\sigma(K) = K$ since $K$ is characteristic in $G$. Therefore the restriction of $\sigma$ to $K$ is an automorphism of $K$. But $H$ is characteristic in $K$, so it is left unchanged by this automorphism, hence $\sigma(H) = H$. Thus every automorphism of $G$ fixes $H$, hence $H$ is characteristic in $G$. 

Homework #9, due 11/4/09 = 4.4.1, 4.4.6, 4.4.8(a)(b)

4.4.1 If $\sigma \in \text{Aut}(G)$ and $\varphi_g$ is conjugation by $g$ prove $\sigma \varphi_g \sigma^{-1} = \varphi_{\sigma(g)}$. Deduce that $\text{Inn}(G) \triangleleft \text{Aut}(G)$.

To show $\sigma \varphi_g \sigma^{-1} = \varphi_{\sigma(g)}$ it is enough to note that, for every $x \in G$,

$$(\sigma \varphi_g \sigma^{-1})(x) = \sigma(\varphi_g(\sigma^{-1}(x)))$$

$$= \sigma(\sigma^{-1}(x)g^{-1})$$

$$= \sigma(g)\sigma(\sigma^{-1}(x))\sigma(g)^{-1}$$

$$= \sigma(g)x\sigma(g)^{-1}$$

$$= \varphi_{\sigma(g)}(x)$$

To prove $\text{Inn}(G) \triangleleft \text{Aut}(G)$ it is enough to show that if an inner automorphism $\varphi_g \in \text{Inn}(G)$ is conjugated by an automorphism $\sigma \in \text{Aut}(G)$, the result is again an inner automorphism of $G$. The calculation above shows that the conjugate of the inner automorphism $\varphi_g$ by an automorphism $\sigma$ is $\varphi_{\sigma(g)}$, which is an inner automorphism. So in fact we do have $\text{Inn}(G) \triangleleft \text{Aut}(G)$.

4.4.6 Prove that characteristic subgroups are normal. Give an example of a normal subgroup that is not characteristic.

Assume $H \text{char } G$. By definition, this means that $\sigma(H) = H$ for every automorphism $\sigma \in \text{Aut}(G)$. In particular, $\varphi_g(H) = H$ for every inner automorphism $\varphi_g \in \text{Inn}(G)$, where $g \in G$ and $\varphi_g(x) = gxg^{-1}$ for every $x \in G$. But the equation $\varphi_g(H) = H$ simply asserts $gHg^{-1} = H$, that is, $H$ is invariant under conjugation by every $g \in G$, so $H$ is normal.

For every group $G$, there is an automorphism $\sigma$ of the direct product $G \times G$ that interchanges the two factors, that is, $\sigma(g, g') = (g', g)$ for all $g, g' \in G$. This automorphism carries the normal subgroup $1 \times G$ to another normal subgroup $1 \times G$. Since $1 \times G \not= G \times 1$, this shows that $G \times 1$ is a normal subgroup that is not characteristic in $G \times G$.

4.4.8(a)(b) Let $G$ be a group with subgroups $H$ and $K$ with $H \leq K$.

(a) Prove that if $H$ is characteristic in $K$ and $K$ is normal in $G$ then $H$ is normal in $G$.

To show $H \triangleleft G$ we must show that $H$ is invariant under any inner automorphism of $G$. Let $g \in G$ and let $\varphi_g$ be conjugation by $g$, the associated inner automorphism of $G$ associated with $g$. Since $K$ is normal in $G$, we have $\varphi_g(K) = K$, so the restriction of $\varphi_g$ to $K$ is an automorphism of $K$. But $H$ is characteristic in $K$, so it is left unchanged by this automorphism of $K$, that is, $\varphi_g(H) = H$, as desired.

(b) Prove that if $H$ is characteristic in $K$ and $K$ is characteristic in $G$ then $H$ is characteristic in $G$. Use this to prove the Klein 4-group $V_4$ is characteristic in $S_4$. 

Let $\sigma$ be an automorphism of $G$. Then $\sigma(K) = K$ since $K$ is characteristic in $G$. Therefore the restriction of $\sigma$ to $K$ is an automorphism of $K$. But $H$ is characteristic in $K$, so it is left unchanged by this automorphism, hence $\sigma(H) = H$. Thus every automorphism of $G$ fixes $H$, hence $H$ is characteristic in $G$. 

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There are four copies of the Klein 4-group $V_4$ inside $S_4$, namely $K_1 := \{(), (12), (34), (12)(34)\}$, $K_2 := \{(), (13), (24), (13)(24)\}$, $K_3 := \{(), (14), (23), (14)(23)\}$, and $K_4 := \{(), (12)(34), (13)(24)\}$. By Prop. 17(4) of 4.4, all automorphisms of $S_n$ are inner automorphisms whenever $n \neq 6$. The inner automorphism of $S_n$ induced by a permutation $\sigma \in S_n$ (conjugation by $\sigma$) is the same as the automorphism induced on $S_n$ by permuting the elements of $\{1, \ldots, n\}$ according to $\sigma$. There are automorphisms of $S_4$ (induced by permutations of the underlying set $\{1, 2, 3, 4\}$) that interchange $K_1, K_2$, and $K_3$. Thus none of $K_1, K_2, K_3$ are characteristic in $S_4$. On the other hand, it is easy to check that every permutation of $\{1, 2, 3, 4\}$ in $S_4$ takes $K_4$ to $K_4$, so $K_4$ is characteristic in $S_4$.

All the elements of $K_4$ are even, so they lie in the subgroup $A_4$ of even permutations of $S_4$. Thus $K_4 \leq A_4 \leq S_4$. $K_4$ is the only subgroup of $A_4$ of order 4, by 3.5.9. Therefore $K_4$ is characteristic in $A_4$. We know $A_4$ is a normal subgroup of $S_4$ (because it has index 2), but $A_4$ is also characteristic in $S_4$ (proof?), so $K_4$ is characteristic in $S_4$. 