Homework #5, due 9/30/09 = 2.2.10, 2.3.24, 2.4.3, 2.5.1

2.2.10 Let $H$ be a subgroup of order 2 in group $G$. Show that $N_G(H) = C_G(H)$.

Deduce that if $N_G(H) = G$ then $H \leq Z(G)$.

If $x \in C_G(H)$ then $xh = hx$ for every $h \in H$. But $xh = hx$ implies $xhx^{-1} = h$, so $xhx^{-1} = h$ for every $h \in H$. This latter statement implies $xHx^{-1} = H$, so $x \in N_G(H)$. Thus we have $N_G(H) \supseteq C_G(H)$.

Now suppose $x \in N_G(H)$. Then $H = xHx^{-1}$. Since $H$ has order 2, $H$ has exactly 2 elements, one of which must be the identity element 1 of $G$. Let the other element of $H$ be $h$, so that $H = \{1, h\}$. From $H = xHx^{-1}$ we get $\{1, h\} = x\{1, h\}x^{-1} = \{xx^{-1}, xhx^{-1}\} = \{1, xhx^{-1}\}$. This implies that either $1 = xhx^{-1}$ or $h = xhx^{-1}$.

The former is impossible because if $1 = xhx^{-1}$ then $h = x^{-1}x = 1$, contradicting our assumption that $H$ has two (distinct) elements 1 and $h$. Consequently $h = xhx^{-1}$, which implies $hx = xh$. Clearly $x$ commutes with 1, and $x$ commutes with $h$ as well by this last equation, so $x$ commutes with all (both) elements of $H$, and therefore $x \in C_G(H)$. This shows the opposite inclusion $N_G(H) \subseteq C_G(H)$, so we have $N_G(H) = C_G(H)$.

Suppose $N_G(H) = G$. From what we have proved so far it follows that $C_G(H) = G$, which says that every element of $G$ commutes with every element of $H$. The center $Z(G)$ is the set of elements that commute with every element of $G$. But we just stated that every element of $H$ is such an element, so $H$ is included in the center of $G$, i.e., $H \subseteq Z(G)$. But both $H$ and $Z(G)$ are subgroups of $G$, so this last inclusion implies $H \leq Z(G)$.

2.3.24 Let $G$ be a finite group and let $x \in G$.

(a) Prove that if $g \in N_G(x)$ then $gxy^{-1} = xa$ for some $a \in \mathbb{Z}$.

From the assumption $g \in N_G(x)$ we have $g(x)g^{-1} = x$. But $gxy^{-1} \in N_G(x)$ so $gxy^{-1} \in \langle x \rangle = \{x^a : a \in \mathbb{Z}\}$ so there is some $a \in \mathbb{Z}$ such that $gxy^{-1} = x^a$.

(b) Prove conversely that if $gxy^{-1} = xa$ for some $a \in \mathbb{Z}$ then $g \in N_G(x)$.

Assume $gxy^{-1} = xa$ and $a \in \mathbb{Z}$. Let $b \in \mathbb{Z}$. If $b > 0$ then

$$
g x^b y^{-1} = g \underbrace{x x^{-1} \cdots x^{-1}}_b y^{-1}
$$

$$
= gxy^{-1}gxy^{-1}gxy^{-1} \cdots gxy^{-1}
$$

$$
= (gxy^{-1})^b
$$

$$
= (x^a)^b
$$

$$
= x^{ab}
$$

and if $b < 0$ then

$$
g x^b y^{-1} = g \underbrace{x^{-1} x^{-1} \cdots x^{-1}}_{-b} y^{-1}
$$

$$
= gxy^{-1}gxy^{-1}gxy^{-1} \cdots gxy^{-1} g^{-1}
$$

$$
= (gx^{-1}y^{-1})^{-b} = 1
$$
Therefore $g^x g^{-1} = x^{ab}$ for all $b \in \mathbb{Z}$, so
\[
g(x)g^{-1} = g\{x^b : b \in \mathbb{Z}\}g^{-1} = \{gx^b g^{-1} : b \in \mathbb{Z}\} = \{x^{ab} : b \in \mathbb{Z}\} \subseteq \langle x \rangle.
\]
But $G$ is finite, so $\langle x \rangle$ is finite, and $g(x)g^{-1}$ is a subgroup isomorphic to $\langle x \rangle$, so $g(x)g^{-1}$ and $\langle x \rangle$ must have the same cardinality and hence are equal: $g(x)g^{-1} = \langle x \rangle$, i.e., $g$ is in the normalizer in $G$ of $\langle x \rangle$.

2.4.3 Prove that if $H$ is an abelian subgroup of $G$ then $\langle H, Z(G) \rangle$ is abelian. Give an explicit example of an abelian subgroup $H$ of a group $G$ such that $\langle H, C_G(H) \rangle$ is not abelian.

First note that all the generators of $\langle H, Z(G) \rangle$ commute with each other: suppose $x, y \in H \cup Z(G)$. If $x$ and $y$ are both in $H$ then $xy = yx$ since $H$ is abelian. If either one of them (say $x$) is in the center $Z(G)$, then $x$ commutes with all elements of $G$, including $y$. A proof by induction (not given here) shows that a subgroup generated by elements that commute with each other must be abelian. Thus $\langle H, Z(G) \rangle$ is abelian.

For an example, consider the direct product $H \times K$ of an abelian group $H$ with a non-abelian group $K$. The subgroup $H \times 1 = \{(h, 1) : h \in H\}$ is abelian (by a simple calculation). Every element of the subgroup $1 \times K = \{(1, k) : k \in K\}$ commutes with every element of $H \times 1$, because $(1, k)(h, 1) = (h, k) = (h, 1)(1, k)$ for all $h \in H$ and all $k \in K$. This shows that $1 \times K$ is included in the centralizer $C_G(H \times 1)$ of $H \times 1$. But since $1 \times K$ is not abelian, it follows that neither the centralizer of $H$ in $G$, nor any subgroup containing it, including $\langle H, C_G(H) \rangle$, is not abelian.

2.5.1 Let $H$ and $K$ be subgroups of $G$. Exhibit all possible sublattices which show only $G$, $1$, $H$ and $K$ and their joins and intersections. What distinguishes the different drawings?

There are four possible 2-chains: $1 = H = K < G$, $1 < H = K = G$, $1 = H < K = G$, $1 = K < H = G$; five possible 3-chains: $1 < H < K = G$, $1 = K < H < G$, $1 < H = K < G$, $1 < K < H = G$, $1 = H < K < G$; two possible 4-chains: $1 < H < K < G$, $1 < K < H < G$. The diamond is possible: $1 = H \cap K < H, K < \langle H \cup K \rangle = G$; the diamond with top is possible: $1 = H \cap K < H, K < \langle H \cup K \rangle < G$; the diamond with tail is possible: $1 < H \cap K < H, K < \langle H \cup K \rangle < G$. The six-element lattice consisting of a diamond with top and tail is possible: $1 < H \cap K < H, K < \langle H \cup K \rangle < G$. What distinguishes the different drawings? I have no idea what this question means. Some drawings are of the same lattice, but with different labelings. Perhaps that is what the authors want.