Homework #4, due 9/23/09 = 1.6.10, 1.6.17, 1.7.4, 1.7.17, 2.1.8

1.6.10 Fill in the details in the proof that the symmetric groups $S_\Delta$ and $S_\Omega$ are isomorphic if $|\Delta| = |\Omega|$ as follows: let $\theta : \Delta \rightarrow \Omega$ be a bijection. Define $\varphi : S_\Delta \rightarrow S_\Omega$ by $\varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1}$ for all $\sigma \in S_\Delta$, and prove the following

(a) $\varphi$ is well-defined, that is, if $\sigma$ is a permutation of $\Delta$ then $\theta \circ \sigma \circ \theta^{-1}$ is a permutation of $\Omega$.

Since $\theta$ is a bijection it has an inverse function $\theta^{-1} : \Omega \rightarrow \Delta$ such that $\theta \circ \theta^{-1}$ is the identity function $I_\Omega$ on $\Omega$ and $\theta^{-1} \circ \theta$ is the identity function $I_\Delta$ on $\Delta$. Suppose $\sigma$ is a permutation of $\Delta$. This implies that $\sigma : \Delta \rightarrow \Delta$ so we have the following situation,

$\Omega \xrightarrow{\theta} \Delta \xrightarrow{\sigma} \Delta \xrightarrow{\theta^{-1}} \Omega$

and we may compose $\theta$, $\sigma$, and $\theta^{-1}$ to obtain a function $\varphi(\sigma) = \theta \circ \sigma \circ \theta^{-1} : \Omega \rightarrow \Omega$. Since $\sigma$ is a permutation it has an inverse $\sigma^{-1} : \Delta \rightarrow \Delta$, so

Let $\tau = \theta \circ \sigma^{-1} \circ \theta^{-1}$. Then

$$\varphi(\sigma) \circ \tau = (\theta \circ \sigma \circ \theta^{-1}) \circ (\theta \circ \sigma^{-1} \circ \theta^{-1})$$

$$= \theta \circ \sigma \circ (\theta^{-1} \circ \theta) \circ \sigma^{-1} \circ \theta^{-1}$$

$$= \theta \circ \sigma \circ I_\Delta \circ \sigma^{-1} \circ \theta^{-1}$$

$$= \theta \circ (\sigma \circ \sigma^{-1}) \circ \theta^{-1}$$

$$= \theta \circ \theta^{-1}$$

$$= I_\Delta$$

and

$$\tau \circ \varphi(\sigma) = (\theta \circ \sigma^{-1} \circ \theta^{-1}) \circ (\theta \circ \sigma \circ \theta^{-1})$$

$$= \theta \circ \sigma^{-1} \circ (\theta^{-1} \circ \theta) \circ \sigma \circ \theta^{-1}$$

$$= \theta \circ \sigma^{-1} \circ I_\Delta \circ \sigma \circ \theta^{-1}$$

$$= \theta \circ (\sigma^{-1} \circ \sigma) \circ \theta^{-1}$$

$$= \theta \circ \theta^{-1}$$

$$= I_\Delta$$

Since $\varphi(\sigma)$ has $\tau$ as left and right inverse, it follows that $\varphi(\sigma)$ is a bijection by Prop 0.1.1(3). Since its domain and codomain are the same set $\Omega$, this means that $\varphi(\sigma)$ is a permutation of $\Omega$.

(b) $\varphi$ is a bijection from $S_\Delta$ onto $S_\Omega$ [Find a 2-sided inverse for $\varphi$.]

Define a function $\psi$ on $S_\Omega$ by $\psi(\xi) = \theta^{-1} \circ \xi \circ \theta$ for every $\xi$ in $S_\Omega$. Note that, since $\theta$ is a bijection, its inverse $\theta^{-1}$ is also a bijection, so the argument above in part (a), with $\Delta$ and $\Omega$ interchanged and $\theta$ replaced with $\theta^{-1}$, shows that $\psi(\xi)$ is a...
permutation of \( \Delta \). Thus \( \psi \) maps \( S_\Omega \) into \( S_\Delta \). It turns out to be a 2-sided inverse of \( \varphi \) because, for every \( \sigma \) in \( S_\Delta \),

\[
\psi(\varphi(\sigma)) = \psi(\theta \circ \sigma \circ \theta^{-1}) \\
= \theta^{-1} \circ (\theta \circ \sigma \circ \theta^{-1}) \circ \theta \\
= (\theta^{-1} \circ \theta) \circ \sigma \circ (\theta^{-1} \circ \theta) \\
= I_\Delta \circ \sigma \circ I_\Delta = \sigma
\]

and, for every \( \xi \) in \( S_\Omega \),

\[
\varphi(\psi(\xi)) = \varphi(\theta^{-1} \circ \xi \circ \theta) \\
= \theta \circ (\theta^{-1} \circ \xi \circ \theta) \circ \theta^{-1} \\
= (\theta \circ \theta^{-1}) \circ \xi \circ (\theta \circ \theta^{-1}) \\
= I_\Omega \circ \xi \circ I_\Omega = \xi
\]

Since \( \varphi \) has a 2-sided inverse, it follows that \( \varphi \) is a bijection by Prop 0.1.1(3).

(c) \( \varphi \) is a homomorphism, that is, \( \varphi(\sigma \circ \tau) = \varphi(\sigma) \circ \varphi(\tau) \) for all \( \sigma, \tau \in S_\Omega \). Then

\[
\varphi(\sigma \circ \tau) = \theta \circ (\sigma \circ \tau) \circ \theta^{-1} \\
= \theta \circ \sigma \circ I_\Delta \circ \tau \circ \theta^{-1} \\
= \theta \circ \sigma \circ (\theta^{-1} \circ \theta) \circ \tau \circ \theta^{-1} \\
= (\theta \circ \sigma \circ \theta^{-1}) \circ (\theta \circ \tau \circ \theta^{-1}) \\
= \varphi(\sigma) \circ \varphi(\tau)
\]

so \( \varphi \) is indeed a homomorphism.

Parts (b) and (c) show that \( \varphi \) is a bijection and a homomorphism, i.e., it is an isomorphism.

1.6.17 Let \( G \) be any group. Prove that the map from \( G \) to itself defined by \( g \mapsto g^{-1} \) is a homomorphism if and only if \( G \) is abelian.

Assume \( g \mapsto g^{-1} \) is a homomorphism. Then, for arbitrary elements \( g, h \in G \), we have \( (gh)^{-1} = g^{-1}h^{-1} \) by the homomorphism condition, but in every group we also have \( (gh)^{-1} = h^{-1}g^{-1} \). From these two equations we get \( g^{-1}h^{-1} = h^{-1}g^{-1} \). This last equation holds for all \( g \) and \( h \), so we may apply it also to \( g^{-1}h \) and \( h^{-1}g \), obtaining \( (g^{-1})(h^{-1})^{-1} = (h^{-1})^{-1}(g^{-1})^{-1} \), which simplifies to \( gh = hg \). Thus any two elements of \( G \) commute, and \( G \) is therefore abelian.

Assume \( G \) is abelian. Then the homomorphism condition holds for the inversion map because, for all \( g, h \in G \),

\[
(gh)^{-1} = h^{-1}g^{-1} \\
= g^{-1}h^{-1} \quad G \text{ is abelian}
\]

1.7.4 Let \( G \) be a group acting on a set \( A \) and fix some \( a \in A \). Show that the following sets are subgroups of \( G \).

(a) The kernel \( \{g \in G | \forall a \in A (g \cdot a = a) \} \) of the action.
First we prove that this set is closed under the group operation of $G$. Suppose $g$ and $h$ are in the kernel. Then, for every $a \in A$,

$$(gh) \cdot a = g \cdot (h \cdot a)$$

action axiom

$$= g \cdot a$$

$h$ is in the kernel

$$= a$$

$g$ is in the kernel

so $gh$ is also in the kernel. Next we show the kernel is closed under inverses. Assume $g$ is in the kernel and $a \in A$. Then $g \cdot a = a$. Applying $g^{-1}$ to both sides of this last equation produces $g^{-1} \cdot (g \cdot a) = g^{-1} \cdot a$. By the first action axiom, $(g^{-1}g) \cdot a = g^{-1} \cdot a$, but $g^{-1}g = 1$ and the second action axiom says that $1 \cdot a = a$, so we get $a = g^{-1} \cdot a$. This shows $g^{-1}$ is also in the kernel.

(b) The stabilizer $G_a = \{g \in G | g \cdot a = a\}$ of $a$ in $G$.

For closure, assume $g$ and $h$ are in the stabilizer of $a$ in $G$. Then $g \cdot a = a$ and $h \cdot a = a$, so, by the first action axiom, $(gh) \cdot a = g \cdot (h \cdot a) = g \cdot a = a$. Thus $gh$ is also in the stabilizer of $a$. Furthermore, $g^{-1} \cdot (g \cdot a) = g^{-1} \cdot a$, but $g^{-1} \cdot (g \cdot a) = (g^{-1}g) \cdot a = 1 \cdot a = a$ by both action axioms, so $a = g^{-1} \cdot a$, i.e., $g^{-1}$ is in the kernel of the action. Finally, note that $1$ is in the kernel by the second action axiom ($1 \cdot a = a$ for all $a \in A$).

1.7.17 Let $G$ be a group and let $G$ act on itself by left conjugation, so each $g \in G$ maps $G$ to $G$ by $x \mapsto gxg^{-1}$. For fixed $g \in G$, prove that conjugation by $g$ is an isomorphism from $G$ onto $G$ itself (i.e. is an automorphism of $G$). Deduce that $x$ and $gxg^{-1}$ have the same order for all $x \in G$ and that for any subset $A \subseteq G$, $|A| = |gAg^{-1}|$.

Let $\varphi(x) = gxg^{-1}$ for all $x \in G$. Clearly $\varphi$ maps $G$ into $G$. First we will show $\varphi$ is a homomorphism. The homomorphism condition holds for $\varphi$ because

$$\varphi(xy) = g(xy)g^{-1} = gx(yg^{-1}) = gx(g^{-1}y)g^{-1} = (gxg^{-1})(gyg^{-1}) = \varphi(x)\varphi(y).$$

Next, $\varphi$ is injective, for if $\varphi(x) = \varphi(y)$ then $gxg^{-1} = gyg^{-1}$, so by multiplying this equation of the left of $g^{-1}$ and on the right by $g$ we get $g^{-1}(gxg^{-1})g = g^{-1}(gyg^{-1})g$, hence $(g^{-1}g)x(g^{-1}g) = (g^{-1}g)y(g^{-1}g)$, hence $1x1 = 1y1$, hence $x = y$. Finally, $\varphi$ is surjective because if $x \in G$ then $\varphi(y) = x$ where $y = g^{-1}xg$, since $\varphi(y) = \varphi(g^{-1}xy) = g(xg^{-1}y)g^{-1} = x$.

Isomorphisms preserve order, so $|x| = |\varphi(x)| = |gxg^{-1}|$ for all $x \in G$. An isomorphism is a bijection, and bijections preserve cardinality of sets, so for any subset $A \subseteq G$, $|A| = |gAg^{-1}|$.

2.1.8 Let $H$ and $K$ be subgroups of $G$. Prove that $H \cup K$ is a subgroup of $G$ if and only if either $H \subseteq K$ or $K \subseteq H$.

First the easy direction: assume that either $H \subseteq K$ or $K \subseteq H$. If the former, then $H \cup K = H$ so $H \cup K$ is a subgroup of $G$ simply because $H$ is already a subgroup of $G$. Similarly, if $K \subseteq H$ then $H \cup K = K$ and again $H \cup K$ is a subgroup of $G$.

Now for the interesting direction. Assume that $H \cup K$ is a subgroup of $G$. It will suffice to assume that neither $H \subseteq K$ nor $K \subseteq H$ and derive a contradiction. Since it is not the case that $K \subseteq H$, there must be some element $k \in K$ such that $k \notin H$. Similarly, since $H \not\subseteq K$, there must be some element $h \in H$ such that $h \notin K$. 
Consider the element $hk$. We have $h \in H \subseteq H \cup K$ and $k \in K \subseteq H \cup K$, but $H \cup K$ is assumed to be a subgroup, so $H \cup K$ is closed under the group operation of $G$. Therefore $hk \in H \cup K$. This implies that either $hk \in H$ or $hk \in K$, but we'll get a contradiction in either case. Note that since $H$ and $K$ are subgroups, we have $h^{-1} \in H$ and $k^{-1} \in K$. If $hk \in H$ then $k = h^{-1}(hk) \in H$, contradicting $k \notin K$, while if $hk \in K$ then $(hk)k^{-1} \in K$, contradicting $h \notin K$. 