Discharging and List coloring

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Outline

1. List coloring
   - From Coloring to List Coloring
   - Coloring vs. List Coloring

2. Discharging
   - What is discharging?
   - Example
Graph Coloring

Definition
The coloring is assignment a color to every vertex.

Definition
The proper coloring is a coloring where adjacent vertices have different colors.

Definition
The chromatic number of graph is minimal number of colors needed by a proper coloring. Denoted by $\chi(G)$. 

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Generalizing The Graph Coloring

- Coloring: All vertices have *same* list of possible colors.
- List coloring: Every vertex has it's own list of possible colors $L(v)$.

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The **list coloring** is assignment colors to the vertices from their own lists. Formally $c : v \rightarrow L(v)$
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k-Choosable And Choosability

Definition

The graph is \textit{k-choosable} if: Size of every color list is \( \geq k \rightarrow \) there is a proper list coloring.

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Choosability of graph \( G \) is minimal \( k \) such that \( G \) is \( k-\)choosable. Denoted by \( \chi_\ell(G) \).
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Relationship Between Chromatic Number And Choosability

- $\chi(G) \leq \chi_\ell(G)$
- $\chi(G) \leq \Delta(G) + 1$ and also $\chi_\ell(G) \leq \Delta(G) + 1$
- Exists graph $G$: $\chi(G) < \chi_\ell(G)$
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Known Theorems:

- Every planar graph is 5-choosable. (all cycles)
- Every planar graph without triangles is 4-choosable. (no 3)
- Every planar bipartite graph is 3-choosable. (no 3, 5, 7, 9, 11, ...)
- There is a non 4-choosable planar graph without triangles.

Problem

Which planar graphs without triangles are 3-choosable?
Known Theorems:

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*Which planar graphs without triangles are 3-choosable?*
The Idea Of Discharging

- Take an imaginary planar counterexample.
- Remove reducible pieces while keeping the planarity.
- Assign weights to vertices and faces.
- Move weights if needed and make all weights $\geq 0$.
- So the reduced graph is not planar since for all planar graphs holds $\sum weigh < 0$.  

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Degree Of Vertices And Faces

- **Vertex** \( v \):
  \[ \text{deg} \ v = |\{ \text{incident edges} \}|. \]

- **Face** \( f \):
  \[ \text{deg} \ f = |\{ \text{incident edge sides} \}|. \]

\[
2|E| = \sum \text{deg} \ v
\]
\[
2|E| = \sum \text{deg} \ f
\]
How To Get The Weights

Start from Euler formula for connected graph:

\[ |E| = |V| + |F| - 2 \]

\[ 2 \times 2|E| + 2|E| = 6|V| + 6|F| - 12 \]

\[ \sum 2 \deg v + \sum \deg f = 6|V| + 6|F| - 12 \]

\[ \sum (2 \deg v - 6) + \sum (\deg f - 6) = -12 \]

Definition

Weights \( w(v) = (2 \deg v - 6), \ w(f) = (\deg f - 6) \)

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Theorem (1)

*Every planar graph without triangles, 4-cycles and 5-cycles is 3-choosable.*
The Reduction Part

Removing things without any effect for 3-choosability.

- Remove vertices of degree 1.
- Remove vertices of degree 2.

We end with a planar graph without triangles, 4-cycles and 5-cycles and minimal vertex degree is 3.
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## Counting Weights

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- $\text{deg}(v) \geq 3 \rightarrow w(v) \geq 0$
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All weights are non-negative.

\[ \sum w(v) + \sum w(f) \geq 0 \]

But for planar graph must hold

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Theorem (2)

Every planar graph without triangles, 5-cycles and adjacent 4-cycles is 3-choosable.
The Reduction Part

Removing things without any effect for 3-choosability.

- Remove vertices of degree 1.
- Remove vertices of degree 2.
- Remove 4-cycles with all vertices of degree 3.

We end with a planar graph without triangles, and 5-cycles, every 4-cycle has vertex \( v : \deg(v) \geq 4 \) and minimal vertex degree is 3.
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We have problems with 4-faces. The weight is $-2$. 

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Dealing with 4-faces

Removing things without any effect for 3-choosability.

- Every 4-face $f$ has its own vertex $v$ with $\deg(v) \geq 4$ and $w(v) \geq 2$.
- Reassign weights:
  - $w'(v) = w(v) - 2$
  - $w'(f) = w(f) + 2$.
- So $w'(f) \geq 0$ and $w'(v) \geq 0$ and sum of all weights is same.

For our graph holds $\sum w(v) + \sum w(f) \geq 0$
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We introduced the list coloring as a generalization of graph coloring.

We described basics of the discharging method.

We proved an example from list coloring.
Problem

*Is there a non 3-choosable graph without triangles and 5 cycles?*

Problem

*What if we allow 4 cycles to share a vertex but not edge?*
Questions?