Characterizations of Strongly Regular Graphs:
Part II: Bose-Mesner algebras of graphs

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Notation

$K$: one of the fields $\mathbb{R}$ or $\mathbb{C}$

$X$: a nonempty finite set

$\text{Mat}_X(K)$: the $K$-algebra consisting of the matrices over $K$, and whose rows and columns are indexed by $X$

$K^X$: the $K$-vector space of column vectors over $K$, and whose coordinates are indexed by $X$

Observe $\text{Mat}_X(K)$ acts on $K^X$ by left multiplication.
We endow $\mathbb{R}^X$ with the dot product
\[
\langle u, v \rangle = u^t v \quad (u, v \in \mathbb{R}^X),
\]
and we endow $\mathbb{C}^X$ with the Hermitean dot product
\[
\langle u, v \rangle = u^t \overline{v} \quad (u, v \in \mathbb{C}^X).
\]

For each $x$ in $X$, let $\hat{x}$ denote the vector in $K^X$ with a 1 in coordinate $x$, and zeros in all other coordinates.

Observe that $\{\hat{x} : x \in X\}$ is an orthonormal basis for $K^X$. 
Let $\Gamma = (X, R)$ denote a graph.

The binary relation on $X$ of being connected by a path is an equivalence relation; the equivalence classes are known as the connected components of $\Gamma$.

$\Gamma$ is said to be connected whenever $X$ is a connected component.

Assume $\Gamma$ is connected, and pick $x, y \in X$. By the distance from $x$ to $y$, we mean the scalar

$$\partial(x, y) = \min\{l : \exists \text{ path of length } l \text{ from } x \text{ to } y\}.$$ 

By the diameter of $\Gamma$, we mean the scalar

$$D := \max\{\partial(x, y) : x, y \in X\}.$$
Let $\Gamma$ be a finite simple (undirected) connected graph.

The **adjacency matrix** for $\Gamma$ is the matrix $A \in \text{Mat}_X(\mathbb{C})$ with $xy$ entry

$$A_{xy} = \begin{cases} 
1 & \text{if } \{x, y\} \in R \\
0 & \text{if } \{x, y\} \notin R
\end{cases} \quad (\forall x, y \in X).$$

The **Bose-Mesner algebra** of $\Gamma$ over $K$, is the subalgebra $M$ of $\text{Mat}_X(K)$ generated by the adjacency matrix $A$ of $\Gamma$.

Observe that:

the dimension of $M$ as a $K$-vector space

= the degree of the minimal polynomial of $A$

= the number of distinct eigenvalues of $\Gamma$. 
Lemma. Let $\Gamma$ be a connected graph with diameter $D$. Let $M$ denote the Bose-Mesner algebra of $\Gamma$ over $K$.

(i) The dimension of $M$ as a $K$-vector space is at least $D + 1$.

(ii) $\Gamma$ has at least $D + 1$ distinct eigenvalues.
Let $\Gamma = (X, R)$ denote a connected graph with diameter $D$. For each integer $i$ ($0 \leq i \leq D$), let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{C})$ with $x, y$ entry

$$(A_i)_{xy} = \begin{cases} 
1 & \text{if } \partial(x, y) = i \\
0 & \text{if } \partial(x, y) \neq i 
\end{cases} \quad (\forall x, y \in X).$$

We call $A_i$ the $i$th distance matrix for $\Gamma$. 
Let $\Gamma = (X, R)$ denote a connected graph with diameter $D$, and distance matrices $A_0, A_1, \ldots, A_D$.

(i) $A_0, A_1, \ldots, A_D$ are linearly independent.

(ii) For all $x \in X$, and for all integers $i$ ($0 \leq i \leq D$),

$$A_i \hat{x} = \sum_{\begin{subarray}{c} y \in X \\ \partial(x, y) = i \end{subarray}} \hat{y}.$$
Lemma. Let $\Gamma = (X, R)$ be a connected graph with diameter $D$, and distance matrices $A_0, A_1, \ldots, A_D$.

(i) $A_0 = I$.

(ii) $A_1 = A$.

(iii) $A_0 + A_1 + \cdots + A_D = J$ (the all 1s matrix).

(iv) $A^t_i = A_i$ \hspace{1cm} $(0 \leq i \leq D)$. 
Let $\Gamma = (X, R)$ denote a connected graph with diameter $D$. 
$\Gamma$ is said to be **distance-regular** whenever for all integers $i$, $0 \leq i \leq D$ and for all $x, y \in X$ at distance $\partial(x, y) = i$, the scalars

$$c_i := |\{z \in X : \partial(x, z) = i - 1, \partial(y, z) = 1\}|,$$

$$a_i := |\{z \in X : \partial(x, z) = i, \partial(y, z) = 1\}|,$$

$$b_i := |\{z \in X : \partial(x, z) = i + 1, \partial(y, z) = 1\}|$$

are constants that are independent of $x$ and $y$.

We refer to the $c_i, a_i, b_i$ as the **intersection numbers** of $\Gamma$. 
**Lemma** Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D$.

(a) $c_0 = 0$, $a_0 = 0$, $b_D = 0$, $c_1 = 1$, and that

$$c_i > 0 \quad (1 \leq i \leq D), \quad b_i > 0 \quad (0 \leq i \leq D - 1).$$

(b) $\Gamma$ is regular with valency $k := b_0$.

(c) $k = c_i + a_i + b_i \quad (0 \leq i \leq D)$. 
Distance-regular graphs with diameter \( D \)

The **Hamming graph** \( H(D, N), (N \geq 2) \).

\[ X = \text{the set of } D\text{-tuples of elements from the set } \{1, 2, \ldots, N\}, \]

\[ R = \{xy : x, y \in X, \text{ } x \text{ and } y \text{ differ in exactly one coordinate}\}. \]

Here

\[ c_i = (N - 1)i, \quad b_i = (D - i)(N - 1) \quad (0 \leq i \leq D). \]

\( H(D, 2) \) is called the **\( D \)-cube**.
The Johnson graph $J(D, N)$, $(N \geq 2D)$.

$X = \text{the set of } D\text{-element subsets of } \{1, 2, \ldots, N\}$,

$R = \{xy : x, y \in X, \ |x \cap y| = D - 1\}.$

Here

$c_i = i^2, \quad b_i = (D - i)(N - D - i) \quad (0 \leq i \leq D).$
The $q$-Johnson graph $J_q(D, N)$, $(N \geq 2D)$.

Let $V$ denote an $N$-dimensional vector space over a finite field $GF(q)$.

\[ X = \text{the set of } D\text{-dimensional subspaces of } V. \]

\[ R = \{xy : x, y \in X, \dim(x \cap y) = D - 1\}. \]

Here

\[ c_i = \left(\frac{q^i - 1}{q - 1}\right)^2 \quad (0 \leq i \leq D), \]

\[ b_i = \frac{q^{2i+1}(q^{D-i} - 1)(q^{N-D-i} - 1)}{(q - 1)^2} \quad (0 \leq i \leq D). \]
Lemma. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D$. Then the distance matrices satisfy

$$AA_i = c_{i+1}A_{i+1} + a_iA_i + b_{i-1}A_{i-1} \quad (1 \leq i \leq D),$$

where $A_{D+1} = 0$, and for convenience $c_{D+1} = 1$. 
Lemma. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D$. Let $M$ denote the Bose-Mesner algebra of $\Gamma$ over $K$.

(i) The distance matrices $A_0, A_1, \ldots, A_D$ form a basis for $M$.

(ii) The dimension of $M$ as a $K$-vector space equals $D + 1$.

(iii) $\Gamma$ has exactly $D + 1$ distinct eigenvalues.
How to compute the eigenvalues of a distance-regular graph from the intersection numbers?

**Def.** Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D$. We define the polynomials $f_0, f_1, \ldots, f_{D+1} \in \mathbb{C}[\lambda]$ by

$$f_0 = 1, \hspace{1cm} f_1 = \lambda,$$

$$\lambda f_i = c_{i+1} f_{i+1} + a_i f_i + b_{i-1} f_{i-1} \quad (1 \leq i \leq D),$$

where for convenience we set $c_{D+1} = 1$. 
Lemma. Let the polynomials $f_0, f_1, \ldots, f_{D+1}$ be as in the above definition.

(i) $\deg f_i = i$ \hspace{1cm} (0 ≤ i ≤ D + 1).

(ii) The coefficient of $\lambda^i$ in $f_i$ is $(c_1 c_2 \cdots c_i)^{-1}$ \hspace{1cm} (0 ≤ i ≤ D + 1).

(iii) $f_i(A) = A_i$ \hspace{1cm} (0 ≤ i ≤ D + 1).

\hspace{1cm} In particular, $f_{D+1}(A) = 0$.

(iv) The distinct eigenvalues of $\Gamma$ are precisely the zeros of $f_{D+1}$.
We now wish to expand our notion of an intersection number.

Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D$. Since the distance matrices $A_0, A_1, \ldots, A_D$ are a basis for the Bose-Mesner algebra, there exist scalars $p^h_{ij} \in \mathbb{C}$ $(0 \leq h, i, j \leq D)$ such that

$$A_iA_j = \sum_{h=0}^{D} p^h_{ij} A_h \quad (0 \leq i, j \leq D).$$

Since $A_i$ and $A_j$ commute,

$$p^h_{ij} = p^h_{ji} \quad (0 \leq h, i, j \leq D).$$
We find that for all integers $h, i, j$ ($0 \leq h, i, j \leq D$), and for all $x, y \in X$ such that $\partial(x, y) = h$,

$$p_{ij}^h = |\{z \in X \mid \partial(x, z) = i, \partial(y, z) = j\}|.$$

In particular, each $p_{ij}^h$ is a nonnegative integer. Comparing the definitions, we see that

$$c_i = p_{i-1,1}^i \quad (1 \leq i \leq D),$$

$$a_i = p_{i1}^i \quad (0 \leq i \leq D),$$

$$b_i = p_{i+1,1}^i \quad (0 \leq i \leq D - 1).$$

In view of this, we often refer to any $p_{ij}^h$ as an intersection number of $\Gamma$. 
It turns out that all of the $p_{ij}^h$ can be computed from $b_0, b_1, \ldots, b_{D-1}, c_1, c_2, \ldots, c_D$.

Set

$$k_i := p_{ii}^0 \quad (0 \leq i \leq D).$$

With $h = 0$, $i = j$, $x = y$, we find that for all integers $i$ ($0 \leq i \leq D$), and for all $x \in X$,

$$k_i = |\{z \in X : \partial(x, z) = i\}|.$$

In particular,

$$k_0 = 1, \quad k_1 = k,$$

and

$$k_i \neq 0 \quad (0 \leq i \leq D).$$

We refer to $k_i$ as the $i$th valency of $\Gamma$. 
Lemma. Let $\Gamma = (X, R)$ be a distance-regular graph with diameter $D$. Then

(i) $k_hp_{ij}^h = k_ip_{ij}^i = k_jp_{ih}^j$ \quad (0 \leq h, i, j \leq D).

(ii) $k_{i-1}b_{i-1} = k_ic_i$ \quad (1 \leq i \leq D).

(iii) $k_i = \frac{b_0b_1\cdots b_{i-1}}{c_1c_2\cdots c_i}$ \quad (0 \leq i \leq D).
Lemma. Let $\Gamma = (X, R)$ denote a distance-regular graph with diameter $D$. Then for all integers $h, i, j$ ($0 \leq h, i, j \leq D$),

(i) $p_{ij}^h = 0$ if one of $h, i, j$ is greater than the sum of the other two;

(ii) $p_{ij}^h \neq 0$ if one of $h, i, j$ is equal to the sum of the other two.
In the sequel, we hope to continue to study on distance-regular graphs. However, (considering the schedule having been booked up for the semester,) next time it is illuminating to consider a slightly more general object called a **commutative association scheme**. For these schemes one still has intersection numbers $p^h_{ij}$; the distance regular graphs correspond to the case where the intersection numbers satisfy (i) and (ii) in the last lemma.