Matrix Completion Problems for Pairs of Related Classes of Matrices

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Abstract For a class $X$ of real matrices, a list of positions in an $n \times n$ matrix (a pattern) is said to have $X$-completion if every partial $X$-matrix that specifies exactly these positions can be completed to an $X$-matrix.

If $X$ and $X_0$ are classes that satisfy the conditions

1. any partial $X$-matrix is a partial $X_0$-matrix,
2. for any $X_0$-matrix $A$ and $\epsilon > 0$, $A + \epsilon I$ is a $X$-matrix, and
3. for any partial $X$-matrix $A$, there exists $\delta > 0$ such that $A - \delta \tilde{I}$ is a partial $X$-matrix (where $\tilde{I}$ is the partial identity matrix specifying the same pattern as $A$) then any pattern that has $X_0$-completion must also have $X$-completion.

However, there are usually patterns that have $X$-completion that fail to have $X_0$-completion.

This result applies to many pairs of subclasses of $P$- and $P_0$-matrices defined by the same restriction on entries, including the classes $P/P_0$-matrices, (weakly) sign-symmetric $P/P_0$-matrices, and non-negative $P/P_0$-matrices. It also applies to other related pairs of subclasses of $P_0$-matrices, such as the pairs classes of $P/P_0,1$-matrices, (weakly) sign-symmetric $P/P_0,1$-matrices and non-negative $P/P_0,1$-matrices.

Furthermore, any pattern that has (weakly sign-symmetric, sign-symmetric, non-negative) $P_0$-completion must also have (weakly sign-symmetric, sign-symmetric, non-negative) $P_0,1$-completion, although these pairs of classes do not satisfy condition (3).

Similarly, the class of inverse $M$-matrices and its topological closure do not satisfy condition (3), but the conclusion remains true, and the matrix completion problem for the topological closure of the class of inverse $M$-matrices is solved for patterns containing the diagonal.

1. Introduction

A partial matrix is a matrix in which some entries are specified and others are not (both no entries specified and all entries specified are also allowed). A completion of a partial matrix is a matrix obtained by choosing values for the unspecified entries. A pattern for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $\{1,...,n\} \times \{1,...,n\}$. A partial matrix specifies the pattern if its specified entries are exactly those listed in the pattern. Note that in this paper a pattern does not need to include all diagonal positions. For a class $X$ of real matrices, we say a pattern has $X$-completion if every partial $X$-matrix specifying the pattern can be completed to an $X$-matrix. The matrix completion problem (for patterns) for the class of $X$-matrices is to determine which patterns have $X$-completion.

Applications of matrix completion problems arise in situations where some data are known but other data are not available, and it is known that the full data matrix must have a certain property. Examples include geophysical problems such as seismic reconstruction problems and electrical and
computer engineering problems including data transmission, coding, and image enhancement. Matrix completion problems also arise in optimization and the study of Euclidean distance matrices.

Matrix completion problems have been studied for many classes of matrices, including positive definite matrices [GJSW], P-matrices [JK], [DH], P₀-matrices [CDHMW], M-matrices [Ho2], M₀-matrices [Ho4], inverse M-matrices [JS1], [Ho1], [Ho3] and many other subclasses of P- and P₀-matrices [FJTU], [Ho4]. A variety of techniques have been developed that apply to matrix completion problems for many classes. In this paper, we examine the specific relationship between the solutions to the matrix completion problems for certain pairs of classes.

The answer to the X-matrix completion problem obviously depends on the definition of a partial X-matrix. For many classes X of matrices, in order for it to be possible to have a completion of a partial matrix to an X-matrix, certain obviously necessary conditions must be satisfied. Such obviously necessary conditions are frequently taken as the definition of a partial X-matrix, as we do here.

For α a subset of \{1,...,n\}, the principal submatrix \(A(\alpha)\) is obtained from the \(n \times n\) matrix A by deleting all rows and columns not in \(\alpha\). For all of the classes X of matrices discussed in this paper, membership in the class is inherited by principal submatrices. Thus in order for a partial X-matrix to have a completion to an X-matrix, it is certainly necessary that every fully specified principal submatrix be an X-matrix. For some classes this is sufficient to define a partial X-matrix. Other classes have entry sign patterns (e.g., all entries are non-negative), so any specified entries must also satisfy the sign pattern. Explicit definitions of a partial matrix for the classes discussed are given in Table 1 and Section 3.

A principal minor of A is the determinant of a principal submatrix of A. The matrix \(A \in \mathbb{R}^{n \times n}\) is a P- (respectively, P₀-, P₀,1-) matrix if every principal minor is positive (non-negative, non-negative and all diagonal elements of A are positive). We examine the relationship between the solutions to the matrix completion problems for pairs of related subclasses of P₀-matrices. If X and Y are classes of matrices with X \(\subseteq\) Y, in general it is not possible to conclude either that a pattern that has Y completion must have X completion (because the completion to a Y-matrix may not be an X-matrix) or that a pattern that has X completion must have Y completion (because there may be a partial Y-matrix that is not a partial X-matrix). However, in cases where there is a natural relationship between the classes X and Y, it is sometimes possible to conclude that any pattern that has Y-completion has X-completion. The pair of related classes may be defined by the same entry restriction (see Table 1 below) on the classes of P- and P₀-matrices, on the classes P- and P₀,1-matrices, or on the classes P₀,1- and P₀-matrices. These pairs of classes are studied in Section 2.

Alternatively, the pair may be a class and its topological closure. For a class X of matrices, the matrix A is in the topological closure of X if A is the limit of a sequence \(A_n\) of matrices in X. The determinant is a continuous function of the entries, so any matrix in the topological closure of the class of P-matrices is a P₀-matrix. If A is a P₀-matrix and \(\varepsilon > 0\), then \(A + \varepsilon I\) is a P₀-matrix [HJ2], so any P₀-matrix is the limit of P-matrices. Thus the class of P₀-matrices is the topological closure of the class of P-matrices. The matrix completion problem for the topological closure of the inverse M-matrices is solved in Section 3 for patterns that include all diagonal positions.

In all these cases it is established that if a pattern has completion for the larger class in the pair then it has completion for smaller class in the pair, and that there is a pattern that has completion for the smaller class in the pair that does not have completion for the larger class in the pair.

2. Pairs of \(\Pi/\Pi_0\)-classes

2.1 Definition The classes of matrices X and X₀ are referred to as a pair of \(\Pi/\Pi_0\)-classes if

1. Any partial X-matrix is a partial X₀-matrix.
2. For any X₀-matrix A and \(\varepsilon > 0\), \(A + \varepsilon I\) is a X-matrix.
3. For any partial X-matrix $A$, there exists $\delta > 0$ such that $A - \delta \bar{I}$ is a partial X-matrix (where $\bar{I}$ is the partial identity matrix specifying the same pattern as $A$).

For any partial P-matrix $A$, there exists $\delta > 0$ such that $A - \delta \bar{I}$ is a partial P-matrix (where $\bar{I}$ is the partial identity matrix specifying the same pattern as $A$), because the determinant is a continuous function of the entries of the matrix. Hence the classes P-matrices and $P_0$-matrices are a pair of $\Pi/\Pi_0$-classes.

Table 1 provides definitions of various pairs of subclasses of P- and $P_0$-matrices and partial matrices for these classes (cf. [Ho4]). The pairs $X/X_0$ of classes listed in Table 1 are “natural” in the sense that the class of $X$-matrices is a subclass of P-matrices and the class of $X_0$-matrices is the analogous subclass of $P_0$-matrices. These pairs are all pairs of $\Pi/\Pi_0$-classes (statement 1 is obvious and statements 2 and 3 follow from statements 2 and 3 for P- and $P_0$-matrices).

<table>
<thead>
<tr>
<th>Class $X/X_0$</th>
<th>Definition of a $X/X_0$-matrix $A$: $A$ is a $P/P_0$ - matrix and whenever the listed entries are specified.</th>
</tr>
</thead>
<tbody>
<tr>
<td>weakly sign-symmetric $P/P_0$-matrices</td>
<td>$a_{ij} a_{ji} \geq 0$ for each $i,j$</td>
</tr>
<tr>
<td>sign-symmetric $P/P_0$-matrices</td>
<td>$a_{ij} a_{ji} &gt; 0$ or $a_{ij} = 0 = a_{ji}$ for each $i,j$</td>
</tr>
<tr>
<td>non-negative $P/P_0$-matrices</td>
<td>$a_{ij} \geq 0$ for all $i,j$</td>
</tr>
<tr>
<td>$M/M_0$-matrices</td>
<td>$a_{ij} \leq 0$ for all $i \neq j$</td>
</tr>
<tr>
<td>symmetric $M/M_0$-matrices</td>
<td>symmetric and $a_{ij} \leq 0$ for all $i \neq j$</td>
</tr>
<tr>
<td>positive definite/semidefinite</td>
<td>symmetric</td>
</tr>
</tbody>
</table>

2.2 Theorem For a pair of $\Pi/\Pi_0$-classes, if a pattern has $\Pi_0$-completion then it must also have $\Pi$-completion. Proof: Let $Q$ be a pattern that has $\Pi_0$-completion, and let $A$ be a partial $\Pi$-matrix specifying $Q$. Let $\bar{I}$ be the partial identity matrix specifying the pattern $Q$. There is a $\delta > 0$ such that $B = A - \delta \bar{I}$ is a partial $\Pi$-matrix, and hence a partial $\Pi_0$-matrix. Since $Q$ has $\Pi_0$-completion, $B$ can be completed to a $\Pi_0$-matrix $\hat{B}$. Then $\hat{A} = \hat{B} + \delta I$ is a $\Pi$-matrix that completes $A$. Thus $Q$ has $\Pi$-completion.

2.3 Corollary
- Any pattern that has $P_0$-completion has $P$-completion.
- Any pattern that has weakly sign-symmetric $P_0$-completion has weakly sign-symmetric $P$-completion.
- Any pattern that has sign-symmetric $P_0$-completion has sign-symmetric $P$-completion.
- Any pattern that has non-negative $P_0$-completion has non-negative $P$-completion.
- Any pattern that has $M_0$-completion has $M$-completion.
- Any pattern that has symmetric $M_0$-completion has symmetric $M$-completion.
- Any pattern that has positive semidefinite-completion has positive definite-completion.
For $\Pi/\Pi_0$ the classes of $P/P_0$-matrices, weakly sign-symmetric $P/P_0$-matrices, and non-negative $P/P_0$-matrices, Corollary 2.3 provides new information. For $\Pi/\Pi_0$ the classes of positive definite/semidefinite matrices and (symmetric) $M/M_0$-matrices, Corollary 2.3 does not provide any new information, because these completion problems have already been solved [GJSW], [JS2], [Ho2], [Ho4], [Ho5]. For $\Pi/\Pi_0$ the classes of sign-symmetric $P/P_0$-matrices, Corollary 2.3 again does not provide any new information, because the sign-symmetric $P_0$-completion problem has been solved and every pattern that has sign-symmetric $P_0$-completion is already known to have sign-symmetric $P$-completion [FJTU], [Ho4].

It is interesting to note that for all the classes in Corollary 2.3, the conclusion is false if the roles of $\Pi$ and $\Pi_0$ are reversed (assuming patterns are allowed to omit some diagonal positions), as the following examples show.

2.4 Example Let $Q_1 = \{(1,1), (1,2), (2,1)\}$. The pattern $Q_1$ has (non-negative, weakly sign-symmetric, sign-symmetric) $P$-completion, (symmetric) $M$-completion and positive definite completion [GJSW], [JK], [Ho2], [Ho4]. The partial matrices $A_{1+} = \begin{bmatrix} 0 & 1 \\ 1 & ? \end{bmatrix}$ and $A_{1-} = \begin{bmatrix} 0 & -1 \\ -1 & ? \end{bmatrix}$ specify $Q_1$. $A_{1+}$ is a partial (weakly sign-symmetric, sign-symmetric, non-negative) $P_0$-matrix and a partial positive semi-definite matrix that cannot be completed to a $P_0$-matrix (and hence cannot be completed to any of the subclasses) because $\det \hat{A}_{1+} = -1$ for any completion $\hat{A}_{1+}$ of $A_{1+}$. $A_{1-}$ is a partial (symmetric) $M_0$-matrix that cannot be completed to a $M_0$-matrix.

Note that pattern $Q_1$ omits a diagonal position. For $\Pi/\Pi_0$ the pair of classes positive definite/semidefinite, $M-/M_0$-, or symmetric $M-/M_0$- matrices, the only patterns that have $\Pi$-completion that do not have $\Pi_0$-completion are patterns that omit diagonal positions (such as $Q_1$) [GJSW], [FJTU], [Ho2], [Ho4], [Ho5]. However, Examples 2.5, 2.6 and 2.7 and Lemma 2.8 below show that this is not the case for pairs $P-/P_0$-, sign-symmetric $P-/P_0$-, weakly sign-symmetric $P-/P_0$-, and non-negative $P-/P_0$-matrices.

2.5 Example Let $Q_2 = \{(1,1), (2,1), (2,2)\}$. The pattern $Q_2$ has sign-symmetric $P$-completion [Ho4, Theorem 3.3]. The partial sign-symmetric $P_0$-matrix $A_2 = \begin{bmatrix} 0 & \, ? \\ 1 & 1 \end{bmatrix}$ specifies $Q_2$ and cannot be completed to a sign-symmetric $P_0$-matrix, because for any sign-symmetric completion $\hat{A}_2$ of $A_2$, $\det \hat{A}_2 < 0$.

2.6 Example The pattern $Q_3 = \{(1,1), (1,2), (2,1) (2,2), (2,3), (3,1), (3,2) (3,3)\}$ has $P$-completion [JK]. The partial $P_0$-matrix $A_3 = \begin{bmatrix} 0 & -1 & ? \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$ specifies $Q_3$ and cannot be completed to a $P_0$-matrix because $\det \hat{A}_3 = -1$ for any completion $\hat{A}_3$ of $A_3$.

2.7 Example Let $Q_4 = \{(1,1), (1,2), (2,2), (2,3), (3,3), (3,4), (4,1), (4,4)\}$. The pattern $Q_4$ has non-negative $P$-completion [Ho4, Corollary 8.5]. The partial non-negative and weakly sign symmetric
$P_0$-matrix $A_4 = \begin{bmatrix} 0 & 1 & ? & ? \\ ? & 0 & 1 & ? \\ ? & ? & 0 & 1 \\ 1 & ? & ? & 0 \end{bmatrix}$ specifies $Q_4$ and cannot be completed to a non-negative $P_0$-matrix nor to a weakly sign symmetric $P_0$-matrix [Ho4, Example 9.8].

**2.8 Lemma** The pattern $Q_4$ has weakly sign symmetric $P$- and weakly sign symmetric $P_{0,1}$-completion.

Proof: Because the class of weakly sign symmetric $P$- ($P_{0,1}$-) matrices is closed under multiplication by positive diagonal matrices, we may assume that all diagonal entries in a partial weakly sign symmetric $P$- ($P_{0,1}$-) matrix are 1. The partial weakly sign symmetric $P_{0,1}$-matrix

$$A_4 = \begin{bmatrix} 1 & a_{12} & ? & ? \\ ? & 1 & a_{23} & ? \\ ? & ? & 1 & a_{34} \\ a_{41} & ? & ? & 1 \end{bmatrix}$$

specifies $Q_4$. If any of $a_{12}$, $a_{23}$, $a_{34}$, $a_{41}$ is 0, then $A$ can be completed to a weakly sign symmetric $P_{0,1}$-matrix by choosing all unspecified entries to be 0. If all of $a_{12}$, $a_{23}$, $a_{34}$, $a_{41}$ are nonzero, then without loss of generality (by use of a diagonal similarity) we may assume that $a_{12} = a_{23} = a_{34} = 1$. If $a_{41} < 0$ then $A$ can be completed to a weakly sign symmetric $P_{0,1}$-matrix by choosing all unspecified entries to be 0. If $a_{41} > 0$ then $A$ can be completed to a weakly sign symmetric $P_{0,1}$-matrix by choosing the 1,3 and 2,4 entries to be 1 and all other unspecified entries to be 0.

It is also possible to apply Theorem 2.2 to other pairs of classes. In particular, it applies to the pair of classes $P$-matrices and $P_{0,1}$-matrices, as well as to pairs of analogously defined subclasses of these classes. To see that these are $\Pi/\Pi_0$-pairs, note that property (1) is clear, (2) follows from property (2) for $P_0$-matrices, and (3) follows property (3) of $P$-matrices.

**2.9 Corollary**

- Any pattern that has $P_{0,1}$-completion also has $P$-completion.
- Any pattern that has weakly sign-symmetric $P_{0,1}$-completion also has weakly sign-symmetric $P$-completion.
- Any pattern that has sign-symmetric $P_{0,1}$-completion also has sign-symmetric $P$-completion.
- Any pattern that has non-negative $P_{0,1}$-completion also has non-negative $P$-completion.

Again, for all the classes in Corollary 2.9, the statement of Corollary 2.9 is false if the roles of $\Pi$ and $\Pi_0$ are reversed (assuming patterns are allowed to omit some diagonal positions), as the following example shows.

**2.10 Example** Let $Q_5 = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2)\}$. The pattern $Q_5$ has (weakly sign-symmetric, sign-symmetric, non-negative) $P$-completion [JK], [Ho4, Theorem 4.6].

The partial (weakly sign-symmetric, sign-symmetric, non-negative) $P_{0,1}$-matrix $A_5 = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & ? \end{bmatrix}$ specifies $Q_5$ and cannot be completed to a $P_{0,1}$-matrix (and hence cannot be completed to any of the subclasses) because $\det \hat{A}_5 = -1$ for any completion $\hat{A}_5$ of $A_5$.  

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The remark about omitting diagonal positions is not necessary for the pair of classes sign-symmetric $P$-$P_{0,1}$-matrices, as the following example shows.

**2.11 Example** Let $Q_6 = \{(1,1), (1,2), (2,1), (2,2), (3,1), (3,2), (3,3)\}$. The pattern $Q_6$ has sign-symmetric $P$-completion [Ho4, Lemma 3.3]. The partial sign-symmetric $P_{0,1}$-matrix

$$A_6 = \begin{bmatrix} 4 & 2 & ? \\ 2 & 1 & ? \\ 4 & -1 & 1 \end{bmatrix}$$

specifies $Q_6$ and cannot be completed to a sign-symmetric $P_{0,1}$-matrix [Ho4, Example 3.4].

Although Theorem 2.2 does not apply to the pair of classes $P_{0,1}$-matrices and $P_0$-matrices (because condition (3) of the definition of pair of $\Pi/\Pi_0$-classes is not true, as the $P_{0,1}$-matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ shows), the conclusion remains true:

**2.12 Theorem**

- Any pattern that has $P_0$-completion also has $P_{0,1}$-completion.
- Any pattern that has weakly sign-symmetric $P_0$-completion also has weakly sign-symmetric $P_{0,1}$-completion.
- Any pattern that has sign-symmetric $P_0$-completion also has sign-symmetric $P_{0,1}$-completion.
- Any pattern that has non-negative $P_0$-completion also has non-negative $P_{0,1}$-completion.

Proof: Let $Q$ be a pattern that has $P_0$-completion, and let $A$ be a partial $P_{0,1}$-matrix specifying $Q$. Clearly $A$ is a partial $P_0$-matrix. Since $Q$ has $P_0$-completion, $A$ can be completed to a $P_0$-matrix $A^\ast$. If $A^\ast$ is not a $P_{0,1}$-matrix, then one or more diagonal entries are 0 (this is the only distinction between a $P_{0,1}$-matrix and a $P_0$-matrix). Since $A$ was a partial $P_{0,1}$-matrix, any diagonal entry specified in $A$ was positive. Let $D = [d_{ij}]$ be defined by $d_{ii} = 0$ if $(i,i) \in Q$, $d_{ii} = 1$ if $(i,i) \notin Q$, and $d_{ij} = 0$ if $i \neq j$. Then $A^\ast + D$ completes $A$ to a $P_{0,1}$-matrix, because it has positive diagonal and, as the sum of a $P_0$-matrix and a non-negative diagonal matrix, $A^\ast + D$ is a $P_0$-matrix. Thus $Q$ has $P_{0,1}$-completion. The same argument works for the weakly sign-symmetric, sign-symmetric and non-negative pairs of subclasses.

Again the statements in Theorem 2.12 are false if the roles of $P_{0,1}$ and $P_0$ are reversed. The Venn diagrams shown in Figure 1 summarize the results established in Corollary 2.9, Theorem 2.12, and the examples. All regions shown in the diagrams are non-empty. The pattern $Q_2$ has sign-symmetric $P_{0,1}$-completion and does not have sign-symmetric $P_0$-completion ([Ho4, Lemma 4.8] and Example 2.5); $Q_3$ has $P_{0,1}$-completion and does not have $P_0$-completion ([Ho4, Lemma 8.1] and Example 2.6); and $Q_4$ has non-negative $P_{0,1}$-completion [Ho4, Corollary 8.5] and weakly sign-symmetric $P_{0,1}$-completion (Lemma 2.8) and does not have either non-negative $P_0$-completion or weakly sign-symmetric $P_0$-completion [Ho4, Example 9.8].
Figure 1: Relationships among the sets of patterns having completion for various classes of matrices.

All the results presented so far inferring $X$-completion of a pattern from $Y$-completion of a pattern involved pairs of classes $X \subseteq Y$. For certain special patterns it is also possible to infer completion of a pattern for the larger class from completion for the smaller class.

A pattern $Q$ is *asymmetric* if $(i,j) \in Q$ implies $(j,i) \notin Q$.

### 2.13 Theorem
- Any asymmetric pattern that has P-completion also has $P_{0,1}$-completion.
- Any asymmetric pattern that has weakly sign-symmetric P-completion also has weakly sign-symmetric $P_{0,1}$-completion.
- Any asymmetric pattern that has sign-symmetric P-completion also has sign-symmetric $P_{0,1}$-completion.
- Any asymmetric pattern that has non-negative P-completion also has non-negative $P_{0,1}$-completion.
Proof: Let $Q$ be an asymmetric pattern that has P-completion, and let $A$ be a partial $P_{0,1}$-matrix specifying $Q$. Since the pattern is asymmetric, there are no fully specified principal submatrices of size larger than 1, and the size 1 matrices are P-matrices. Thus $A$ is a partial P-matrix, and can be completed to a P-matrix $\hat{A}$. Clearly $\hat{A}$ is a $P_{0,1}$-matrix that completes $A$. The same argument works for the weakly sign-symmetric, sign-symmetric and non-negative pairs of subclasses. ■

Since it is known [CDHMW] that every asymmetric pattern has both P-completion and $P_{0}$-completion, either Theorem 2.12 or Theorem 2.13 may be used to conclude that every asymmetric pattern has $P_{0,1}$-completion.

3. The Topological Closure of the Class of Inverse M-matrices

In this section patterns are assumed to contain all diagonal positions.

A matrix is defined to be an inverse M-matrix if it is nonsingular and its inverse is a M-matrix, i.e., a P-matrix with non-positive off-diagonal entries. Any inverse M-matrix is non-negative [HJ2]. Every principal submatrix of an inverse M-matrix is an inverse M-matrix [J1]. A partial matrix is a partial inverse M-matrix if every fully specified principal submatrix is an inverse M-matrix and all specified entries are non-negative. In [JS2], a matrix is defined to be a singular inverse M-matrix if it is singular and in the topological closure of the class of inverse M-matrices. We say the matrix $A$ is TCIM-matrix if it is in the topological closure of the class of inverse M-matrices, that is, if $A$ is the limit of a sequence of inverse M-matrices. Since the entries of $A^{-1}$ are continuous functions of the entries of $A$ (as long as $A$ remains nonsingular) and a nonsingular limit of M-matrices is an M-matrix, any nonsingular TCIM-matrix is an inverse M-matrix. Any TCIM-matrix is non-negative. A matrix is a partial TCIM-matrix if every fully specified principal submatrix is a TCIM-matrix and all specified entries are non-negative.

We examine the relationship between the inverse M-matrix completion problem and the TCIM-matrix completion problem. Because the class of $P_{0}$-matrices is the topological closure of the class of P-matrices, one might hope to apply Theorem 2.2 to the pair of classes inverse M-/TCIM-matrices; however, this pair of classes fails to satisfy the third required property, as the next example shows:

3.1 Example Let $B = \begin{bmatrix} 8 & 3 & 1 \\ 4 & 6 & 2 \\ 4 & 3 & 5 \end{bmatrix}$. Then
\[
(B - \delta I)^{-1} = \frac{1}{\det(B - \delta I)} \begin{bmatrix} 24 - 11\delta + \delta^2 & -12 + 3\delta & \delta \\ -12 + 4\delta & 36 - 13\delta + \delta^2 & -12 + 2\delta \\ -12 + 4\delta & -12 + 3\delta & 36 - 14\delta + \delta^2 \end{bmatrix}.
\]
By choosing $\delta = 0$, it is clear that $B$ is an inverse M-matrix. However, 1,3-entry of $(B - \delta I)^{-1}$ will be positive for $\delta > 0$, so there does not exist $\delta > 0$ such that $B - \delta I$ is an inverse M-matrix.

However, the other two properties in Theorem 2.2 remain true (the first is obvious):

3.2 Lemma [J2] For any TCIM-matrix $A$ and $\varepsilon > 0$, $A + \varepsilon I$ is an inverse M-matrix.
Proof: Suppose $A$ is a TCIM-matrix. Since $A$ is in the closure of the class of inverse M-matrices, there is a sequence $A_n$ of inverse M-matrices with $\lim_{n \to \infty} A_n = A$. Then for $\varepsilon > 0$, $\lim_{n \to \infty} (A_n + \varepsilon I) = A + \varepsilon I$. Since $A_n$ is an inverse M-matrix, so is $A_n + \varepsilon I$, because the sum of an inverse M-matrix and a non-negative diagonal matrix is an inverse M-matrix [J1]. So $A + \varepsilon I$ is in the closure of the inverse M-matrices. But $A$ is a $P_{0}$-matrix, so $A + \varepsilon I$ is a P-matrix, and
hence is nonsingular. Therefore \( A + \varepsilon I \) is an inverse M-matrix.

In addition to the property that a principal submatrix of an inverse M-matrix is an inverse M-matrix, the class of inverse M-matrices possesses two other important properties in the study of matrix completions: it is closed under permutation similarity and direct sums. Hence the class of TCIM-matrices is closed under permutation similarity and direct sums, i.e., if \( A \) and \( B \) are TCIM matrices and \( P \) is a permutation matrix of the same size as \( A \), then \( P^{-1}AP \) and \( A \oplus B \) are TCIM-matrices. (Thus the classes inverse M-matrices and TCIM-matrices are HSP classes as defined in [Ho5]).

In recent years graphs and digraphs have been used very effectively to study matrix completion problems. Here we shall use digraphs to study the matrix completion problem for TCIM-matrices.

A digraph \( G = (V_G,E_G) \) is a finite set of positive integers \( V_G \), whose members are called vertices, and a set \( E_G \) of ordered pairs \((v,u)\) of distinct vertices, called arcs. The order of a digraph is the number of vertices.

The digraph \( G = (V_G,E_G) \) is isomorphic to the digraph \( H = (V_H,E_H) \) by isomorphism \( \phi \) if \( \phi \) is a one-to-one map from \( V_G \) onto \( V_H \) and \((v,w) \in E_G \) if and only if \((\phi(v), \phi(w)) \in E_H \).

A subdigraph of the digraph \( G = (V_G,E_G) \) is a digraph \( H = (V_H,E_H) \), where \( V_H \subseteq V_G \) and \( E_H \subseteq E_G \) (note that \((v,u) \in E_H \) requires \( v,u \in V_H \) since \( H \) is a digraph). If \( W \subseteq V_G \), the subdigraph of \( G \) induced by \( W \), \(<W>\), is the digraph \((W,E_W)\) with \( E_W = E_G \cap (W \times W) \). A subdigraph induced by a subset of vertices is also called an induced subdigraph.

A path (respectively, semipath) in a digraph \( G=(V,E) \) is sequence of vertices \( v_1, v_2, ..., v_{k-1}, v_k \) in \( V \) such that for \( i=1,...,k-1 \) the arc \((v_i,v_{i+1}) \in E \) (respectively, \((v_i,v_{i+1}) \in E \) or \((v_{i+1},v_i) \in E \)) and all vertices are distinct except possibly \( v_1 = v_k \). Clearly, a path is a semipath, although the converse is false. A (semi)path is open if the first and last vertices are distinct. The length of the (semi)path \( v_1, v_2, ..., v_{k-1}, v_k \) is \( k-1 \).

A digraph is connected if there is a semipath from any vertex to any other vertex (a digraph of order 1 is connected); otherwise it is disconnected. A component of a digraph is a maximal connected subdigraph. A digraph is strongly connected if there is a path from any vertex to any other vertex. Clearly, a strongly connected digraph is connected, although the converse is false.

A digraph \( G=(V,E) \) with \( V = \{1, ..., n\} \) is called a source/sink cut bipartite digraph if \( V \) can be partitioned into disjoint sets \( S \) (sources) and \( T \) (sinks), such that for any \((u,v) \in E \), \( u \in S \) and \( v \in T \).

Let \( A \) be a (fully specified) \( n \times n \) matrix. The nonzero-digraph of \( A \) is the digraph having vertex set \( \{1,...,n\} \) and, as arcs, the ordered pairs \((i,j)\) where \( i \neq j \) and \( a_{ij} \neq 0 \). The characteristic matrix of a pattern \( Q \) for \( n \times n \) matrices is the \( n \times n \) matrix \( C \) such that \( c_{ij} = 1 \) if the position \((i,j) \in Q \) and \( c_{ij} = 0 \) if \((i,j) \notin Q \). For a pattern \( Q \) that contains all diagonal positions, the digraph of \( Q \) is the nonzero-digraph of the characteristic matrix of \( Q \).

A digraph \( G=(V,E) \) with \( V = \{1,...,n\} \) is called a pattern-digraph. Clearly such a digraph is the digraph of the pattern for \( n \times n \) matrices \( Q_G = E \cup \{(v,v) : 1 \leq v \leq n\} \). A partial matrix that specifies \( Q \) is referred to as having TCIM-completion if \( Q_G \) does.

A pattern \( Q \) is permutation similar to a pattern \( R \) if there is a permutation \( \pi \) of \( \{1,...,n\} \) such that \( R = \{(\pi(i), \pi(j)) : (i,j) \in Q \} \). Equivalently, \( C_Q \) is permutation similar to \( C_R \). Relabeling the vertices of a digraph diagram, which performs a digraph isomorphism, corresponds to performing a permutation similarity on the pattern. Since the class of TCIM-matrices is closed under permutation similarity, we are free to relabel digraphs as desired.

Note that when using digraphs, patterns are assumed to include all diagonal positions.

**3.3 Lemma** Let \( Q \) be a pattern that has TCIM-completion, and let \( G \) be the digraph of \( Q \). Then every pattern-digraph isomorphic to an induced subdigraph of \( G \) has TCIM-completion.
Proof: Suppose the pattern-digraph $K$ is isomorphic to the induced subdigraph $H$ of $G$ by isomorphism $\phi$. Let $A$ be a partial TCIM-matrix specifying $K$. Define a partial matrix $B$ specifying $Q$ by defining $b_{ij}$ for $(i,j) \in Q$ as follows: If $i, j \in V_H = \phi(V_K)$, $b_{ij} = a_{\phi^{-1}(i) \phi^{-1}(j)}$. If $i \notin V_H$, $b_{ii} = 1$. If $i \neq j$ and $i \notin V_H$ or $j \notin V_H$, $b_{ij} = 0$. Any fully specified submatrix of $B$ is permutation similar to a fully specified submatrix of $A$ or to the direct sum of an identity matrix with such a matrix. Since the class of TCIM-matrices is closed under direct sums and permutation similarity, $B$ is a partial TCIM-matrix. Since $Q$ has TCIM-completion, we can complete $B$ to a TCIM-matrix $\hat{B}$. The principal submatrix $\hat{B}(V_H)$ then is used to define a completion $\hat{A}$ of $A$ by $\hat{a}_{ij} = \hat{b}_{\phi(i)\phi(j)}$.

3.4 Lemma Let $G$ be a pattern-digraph. If every component of $G$ is isomorphic to a pattern-digraph that has TCIM-completion, then $G$ has TCIM-completion.

Proof: Relabel the vertices of $G$ to obtain an isomorphic digraph $G'$ in which the vertices of each component are consecutive numbers. Every component of $G'$ is isomorphic to a component of $G$ and hence to a pattern-digraph that has TCIM-completion. Any partial TCIM-matrix $A$ specifying $G'$ is a block diagonal partial matrix, with each diagonal block corresponding to a component of $G'$. Complete each of these blocks to a TCIM-matrix via the isomorphism to a pattern-digraph that has TCIM-completion, resulting in a partial matrix $B$. Since the class of TCIM-matrices is closed under direct sums, $B$ can be completed to a TCIM-matrix by setting all entries outside the diagonal blocks to 0. Since the class of TCIM-matrices is closed under permutation similarity, any partial TCIM-matrix specifying $G$ can be transformed into a partial TCIM-matrix specifying $G'$, completed to a TCIM-matrix, and transformed back. Thus $G$ has TCIM-completion.

3.5 Lemma The pattern-digraphs shown in Figure 2 do not have TCIM-completion.

Proof: Note that each digraph shown in Figure 2 contains an open path of length 2.

The pattern-digraphs $q=3$ $n=3$, $q=4$, $n=2, 3, 4$ and $q=5$ all include the $(1,2)$, $(2,3)$ and $(1,3)$ positions, as well as the diagonal $(1,1)$, $(2,2)$ and $(3,3)$, and all omit at least one off-diagonal position. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ a & 1 & 1 \\ b & c & 1 \end{bmatrix}$, where each of $a$, $b$, $c$ is either 0 or unspecified with at least one unspecified, so $A$ is a partial TCIM-matrix. When values are chosen for any of $a$, $b$, $c$ to obtain a completion $\hat{A}$, $(\hat{A} + \epsilon I)^{-1} = \frac{1}{\det(\hat{A} + \epsilon I)} \begin{bmatrix} 1 - c + 2 \epsilon + \epsilon^2 & -1 - \epsilon & 1 \\ -a + b - a \epsilon & 1 + 2 \epsilon + \epsilon^2 & -1 - \epsilon \\ -b + ac - b \epsilon & b - c - c \epsilon & 1 - a + 2 \epsilon + \epsilon^2 \end{bmatrix}$. Since $(\hat{A} + \epsilon I)^{-1}|_{13} = \frac{1}{\det(\hat{A} + \epsilon I)} > 0$, $(\hat{A} + \epsilon I)^{-1}$ is not an M-matrix, $(\hat{A} + \epsilon I)$ is not an inverse M-matrix and $\hat{A}$ is not a TCIM matrix. Thus the pattern-digraphs $q=3$ $n=3$, $q=4$, $n=2$, $q=4$, $n=3$, $q=4$ $n=4$ and $q=5$ do not have TCIM-completion.

The pattern-digraphs $q=2$ $n=2$, $q=3$ $n=1, 2, 4$, and $q=4$ $n=2$ all include the $(1,2)$ and $(2,3)$ positions, as well as the diagonal $(1,1)$, $(2,2)$ and $(3,3)$, and all omit $(1,3)$. Let $B = \begin{bmatrix} 1 & 1 & ? \\ a & 0 & 1 \\ b & c & 1 \end{bmatrix}$, where $?$ is unspecified and each of $a$, $b$, $c$ is either 0 or unspecified, so $B$ is a partial TCIM-matrix. If $a$ is
unspecified, any completion \( \hat{B} \) of \( B \) to a TCIM-matrix must set \( a \) to 0, because \( a \) must be chosen non-negative, and \( \det \hat{B}(\{1,2\}) = -a > 0 \), since \( \hat{B} \) must be a \( P_0 \)-matrix. Similarly, \( c \) must be 0.

Thus any completion of \( B \) to a TCIM-matrix must be of the form \( \hat{B}_y = \begin{bmatrix} 1 & 1 & y \\ 0 & 0 & 1 \\ b & 0 & 1 \end{bmatrix} \), where \( y \) (and \( b \) if necessary) is/are specific choice(s) for entry(ies) 1,3 (and 3,1). In order for \( \hat{B}_y \) to be a TCIM-matrix, \( \hat{B}_y + \epsilon I \) must be an inverse M-matrix for every \( \epsilon > 0 \). But

\[
\left( \hat{B}_y + \epsilon I \right)^{-1} = \frac{1}{\det(\hat{B}_y + \epsilon I)} \begin{bmatrix} \epsilon + \epsilon^2 & -1 - \epsilon & 1 - \epsilon y \\ b & 1 + 2\epsilon + \epsilon^2 - by & -1 - \epsilon \\ -b\epsilon & b & \epsilon + \epsilon^2 \end{bmatrix}.
\]

For any \( y \) there is an \( \epsilon_y > 0 \) such that \( 0 < 1 - \epsilon_y \), and so \( (\hat{B}_y + \epsilon I)_{13} = \frac{1 - \epsilon_y y}{(\hat{B}_y + \epsilon I)} > 0 \). Thus \( (\hat{B}_y + \epsilon, I)^{-1} \) is not an M-matrix and \( \hat{B}_y + \epsilon, I \) is not an inverse M-matrix. Thus \( \hat{B}_y \) is not a TCIM-matrix and \( B \) cannot be completed to a TCIM-matrix. Thus the pattern-digraphs \( q=2 \ n=2, q=3 \ n=1, 2, 4, \) and \( q=4 \ n=2 \) do not have TCIM-completion.

![Pattern-digraphs that do not have TCIM-completion](notation from [Ha]).

3.6 Lemma A digraph \( G \) is a source/sink cut bipartite digraph if and only if \( G \) does not contain any path of length 2.

Proof: If \( G = (V,E) \) is a source/sink cut bipartite digraph (with sources \( S \) and sinks \( T \)) and \((u,v) \in E, \) then \( u \in S \) and \( v \in T, \) so \( G \) does not contain any arcs \((u,v)\) and \((v,w)\) and hence does not contain a path \( u, v, w \) of length 2. If \( G \) does not contain any path of length 2, let \( S = \{u: (u,v) \in E\} \) and \( T = E - S. \) If \( (u,v) \in E, \) then there does not exist \((v,w)\) in \( E \) because then \( u, v, w \) would be a path of length 2. Thus \( v \notin S \) so \( v \in T \) and \( G \) is a source/sink cut bipartite digraph.

3.7 Lemma If a connected digraph \( G \) does not contain an induced subdigraph isomorphic any to the digraphs in Figure 2, then \( G \) is a clique or a source/sink cut bipartite digraph.

Proof: Suppose \( G \) is not a source/sink cut bipartite digraph. Then \( G \) contains a path \( \Gamma \) of length 2.

For any path \( u, v, w \) of length 2 in \( G, \) we show that \( <u,v,w> \) is a clique: If \( w = u \) then \( <u,v,w> = <u,v> \) is a clique. If \( w \neq u, \) an examination of the table of digraphs of order 3 in [Ha] shows that the only digraphs of order 3 that contain an open path of length 2 are those in Figure 2 and the clique on 3 vertices, so \( <u,v,w> \) is a clique.
Let $H$ be maximal among cliques in $G$ that contain the clique induced by the path $\Gamma$ of length 2. If $H \neq G$, then since $G$ is connected, there is a vertex $z$ not in $H$ and a vertex $x$ in $H$ such that $(z,x)$ or $(x,z)$ is in $H$. The order of $H$ is greater than or equal to 2, so let $y$ be any other vertex in $H$. Since $H$ is a clique, $(x,y)$ and $(y,x)$ are in $H$. So either $G$ contains both $(z,x)$ and $(x,y)$ or $G$ contains both $(y,x)$ and $(x,z)$, and $x, y$ or $y, x, z$ is a path of length 2. So as before, $(x,y,z)$ is a clique, so $(z,x),(x,z),(y,z),(z,y)$ are in $G$ for vertex $x$ and any other vertex $y$ of $H$. Then $(z,y), (y,z)$ is a clique and $H$ is not maximal, contradicting the choice of $H$. Thus $G$ is a clique.

The zero-completion of a partial matrix is the matrix obtained by setting all unspecified entries to zero.

**3.8 Theorem** A pattern $Q$ that includes all diagonal positions has TCIM-completion if and only if each component of its digraph $G$ is a source/sink cut bipartite digraph or a clique. For such a pattern $Q$, the zero-completion of a partial TCIM-matrix specifying $Q$ is a TCIM-matrix.

**Proof:** Consider first a pattern-digraph $H$ that is a source/sink cut bipartite digraph. By relabeling the vertices, we may assume that $S=\{1,\ldots,k\}$ and $T=\{k+1,\ldots,n\}$. Let $B$ be any non-negative partial matrix specifying $H$ and let $B_0$ be the zero completion of $B$. For any $\varepsilon>0$, $B_0 + \varepsilon I$ can be partitioned as $\begin{bmatrix} D_1 & B_{12} \\ 0 & D_2 \end{bmatrix}$ with $D_1$ and $D_2$ positive diagonal matrices and $B_{12}$ non-negative. From the formula for the inverse of a partitioned matrix, $(B_0 + \varepsilon I)^{-1} = \begin{bmatrix} D_1^{-1} & -D_1^{-1}B_{12}D_2^{-1} \\ 0 & D_2^{-1} \end{bmatrix}$. Thus $(B_0+\varepsilon I)^{-1}$ is an M-matrix, $B_0+\varepsilon I$ is an inverse M-matrix and $B_0$ is a TCIM-matrix. So $H$ has TCIM-completion.

Thus if each component of the digraph $G$ of $Q$ is a source/sink cut bipartite digraph or a clique then each component is isomorphic to a pattern-digraph that has TCIM-completion, so by Lemma 3.4, $G$ (and $Q$) have TCIM-completion. Note that all the completions used involve setting all unspecified entries to 0, i.e., the zero-completion.

For the converse, suppose $Q$ has TCIM-completion and let $G$ be its digraph. Let $H$ be a component of $G$. Any induced subdigraph of $H$ is an induced subdigraph of $G$, so by Lemmas 3.3 and 3.5, $H$ cannot contain an induced subdigraph isomorphic to any of the digraphs in Fig. 2. So by Lemma 3.7, $H$ is a clique or a source/sink cut bipartite digraph.

Even though Theorem 2.2 does not apply, the conclusion of Theorem 2.2 remains true, because a pattern that includes all diagonal positions has inverse M-completion if and only if every strongly connected induced subdigraph is path/cycle-clique [Ho1]. Any clique or source/sink cut bipartite digraph is path/cycle-clique.

**3.9 Corollary** If a pattern that includes all diagonal positions has TCIM-completion then it has inverse M-completion.

Like some of the classes studied in Section 2, there are patterns that include all diagonal positions and have inverse M-completion but do not have TCIM-completion. For example, the pattern whose digraph is $q=2, n=2$ has inverse M-completion [Ho1], but does not have TCIM completion.

**4. Conclusion**

It should be noted that the conclusion of Theorem 2.2 is false for some pairs consisting of a class $X$ and its topological closure. For example, the topological closure of the class of sign-symmetric $P_0$-matrices is the class of weakly sign-symmetric $P_0$-matrices. And the pattern
Q₂ = {(1,1), (2,1), (2,2)} does have weakly sign-symmetric P₀-completion (set the unspecified entry to 0) but does not have sign-symmetric P-completion, as the partial sign-symmetric P₀-matrix

\[ A₂ = \begin{bmatrix} 0 & ? \\ 1 & 1 \end{bmatrix} \] shows.

An important distinction between the pair of classes sign-symmetric P₀-/weakly sign-symmetric P₀-matrices and the pair of classes inverse M-/TCIM-matrices is that in the latter case the first and second conditions of Definition 2.1 hold, but the second condition does not hold in the former. Note also that these two properties were sufficient in the discussion of pairs of subclasses of P₀,₁- and P₀-matrices in Section 2. Properties (1) and (2) of Definition 2.1 appear to be the key ingredients in deducing completion results about one class from another.

References


