Path Covers of Trees

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Abstract Minimal path covers of trees have arisen in combinatorial matrix analysis, where the path cover number is the maximum multiplicity of an eigenvalue in a symmetric matrix whose graph is the tree [2]. We study the structure of minimal path covers and minimal path trees and provide an algorithm for producing a minimal path cover, a characterization of when a minimal path is cover unique, an algorithm for transforming one minimal path cover into another, and a characterization of which path covers are minimal.

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By a path cover \( P \) of a tree \( T \) we mean a set of vertex disjoint induced paths of \( T \) that collectively contains all vertices of \( T \). A single vertex constitutes a path. Path cover \( P \) of \( T \) is minimal if no other path cover of \( T \) has fewer paths, and the path cover number \( P(T) \) is the number of paths in any minimal path cover. Minimal path covers are not generally unique. We will describe a path by an ordered list of its vertices, as in the next example.

Example 1
Let \( T \) be the tree shown in Figure 1. The path cover \( \{(1, 2, 3), (4), (5, 6, 7)\} \) of \( T \) is minimal, but so is \( \{(1, 2, 4, 6, 5), (3), (7)\} \).

\begin{figure}[h]
\centering
\begin{tikzpicture}
  \node (1) at (0,0) {1};
  \node (2) at (1,1) {2};
  \node (3) at (2,0) {3};
  \node (4) at (1,-1) {4};
  \node (5) at (0,-2) {5};
  \node (6) at (1,-2) {6};
  \node (7) at (2,-2) {7};
  \draw (1) -- (2);
  \draw (2) -- (3);
  \draw (2) -- (4);
  \draw (4) -- (5);
  \draw (4) -- (6);
  \draw (6) -- (7);
\end{tikzpicture}
\caption{The tree \( T \)}
\end{figure}

Closely associated with a (minimal) path cover \( P \) of a tree \( T \) is a particular presentation of \( T \) in which each of the paths of \( P \) is drawn horizontally (on different levels) and the remaining edges (called connector edges) are drawn as well. We call such a presentation a (minimal) path tree for \( T \). For example, the presentation of \( T \) in Figure 1 is the minimal path tree associated with the path cover \( \{(1, 2, 3), (4), (5, 6, 7)\} \). Of course, a minimal path tree need not be unique.
The minimal path tree associated with the path cover \{(1, 2, 4, 6, 5), (3), (7)\} of \(T\) in Example 1 is shown in Figure 2. We will denote the (minimal) path tree associated with a (minimal) path cover \(\mathbf{P}\) by \(\mathbf{P}\).

Figure 2 The minimal path tree for minimal path cover \{(1, 2, 4, 6, 5), (3), (7)\}

The purpose of a (minimal) path tree is to geometrically display the roles of both the (minimal) path cover and the connector edges. A path tree may have many different drawings; two drawings of the same path tree are illustrated in Figure 3.

Figure 3 Two drawings of the same path tree

Minimal path covers have arisen in combinatorial matrix analysis, where the path cover number \(P(T)\) is the maximum multiplicity of an eigenvalue in a symmetric matrix whose graph is \(T\) [2]. Our purpose here is to carefully study the structure of minimal path covers and minimal path trees. It may well be that there are other sources of interest besides matricial questions. To this end we address a number of natural questions about minimal path covers and minimal path trees.

1. How may a minimal path cover, and thus the path cover number, be efficiently found?
2. When is a minimal path cover unique?
3. If a minimal path cover is not unique how may it simply be transformed into another minimal path cover?
4. Which path covers are minimal?
5. Which paths must appear in any minimal path cover?
6. Which edges of a tree must be connector edges for any minimal path cover?

We introduce some additional notation: The order of a graph \(G\), denoted \(n(G)\), is the number of vertices in \(G\). The number of paths in path cover \(\mathbf{P}\) is denoted \(|\mathbf{P}|\).
If \( v \) is a vertex of a graph \( G \), we denote by \( G - v \) the subgraph of \( G \) obtained by deleting \( v \) and all edges incident with \( v \) from \( G \). An induced subgraph of \( G \) is a subgraph obtained by deleting some subset \( S \) of vertices (and their incident edges); this is denoted \( G - S \). If \( G \) is a tree, an induced subgraph need not be a tree, because it may be disconnected, but it is a forest (the union of one or more disjoint trees). If \( K \) is an induced subgraph of \( G \), we abuse the notation by writing \( G - K \) for \( G - V(K) \), where \( V(K) \) is the set of vertices of \( K \).

The high degree vertices of tree \( T \) are the vertices of degree 3 or more. The subgraph induced by the set of high degree vertices is denoted \( H \) (or \( H(T) \) if there is any danger of confusion).

Let \( T_1 \) and \( T_2 \) be disjoint trees with \( v_i \) a vertex of \( T_i \). The edge sum of \( T_1 \) and \( T_2 \) at \( v_1 \) and \( v_2 \), \( \sum_{v_1, v_2} T_1 + T_2 \) is the tree obtained from \( T_1 \cup T_2 \) by joining \( v_1 \) and \( v_2 \) with the edge \{\( v_1, v_2 \}\}.

A vertex \( v \) of \( T \) is called central if in every minimal path cover \( v \) is in the interior of the path in which it is contained. Note the property of being central is a property of the vertex independent of a particular minimal path cover.

The following lemma is established in [1].

**Lemma 1** Let \( T = T_1 + T_2 \). Then \( P(T_1) + P(T_2) - 1 \leq P(T) \leq P(T_1) + P(T_2) \) and \( P(T) = P(T_1) + P(T_2) \) if and only if at least one \( v_i \) is central in \( T_i \).

A path \( P \) is a pendent path (of \( v \), in \( T \)) if \( v \) is a high degree vertex of tree \( T \) and \( P \) is a component of \( T - v \). A tree is a generalized star if it has at most one high degree vertex.

A pendent generalized star of a graph \( T \) is an induced subgraph \( S \) of \( T \) such that

1) there is exactly one vertex \( v \) of \( S \) that is a high degree vertex of \( T \)
2) \( T - v \) has \( k+1 \) components and exactly \( k \) of the components \( T - v \) of are pendent paths of \( v \).
3) \( S \) is induced by the vertices of the \( k \) pendent paths and \( v \).

(Pendent) generalized stars play a crucial role in the algorithm for finding a minimal path cover that we shall describe shortly. A generalized star that has no high degree vertex (i.e., a path) is degenerate; otherwise it is nondegenerate. (A pendent generalized star is always nondegenerate, since it always has a vertex of high degree in \( T \)). In a nondegenerate (pendent) generalized star the unique high degree vertex is called the center. In a pendent generalized star \( S \) of \( T \) the unique vertex of \( T - S \) that is a neighbor of the center is called the anchor of \( S \). Figure 4 shows a pendent generalized star \( S \) (shown inside the oval) with center \( v \), anchor \( w \) and 4 pendent paths. Note that \( T \) has been presented as a minimal path tree and two pendent paths have been joined to \( v \) to form one path of this path tree.

**Lemma 2** Any tree \( T \) either contains a pendent generalized star or is a generalized star itself. Proof: If \( T \) has no high degree vertices then \( T \) is a path, i.e., a degenerate generalized star. If \( T \) has at least one high degree vertex, proceed as follows: Define \( T' \) to be the tree obtained from \( T \) by removing all pendent paths. If \( n(T') = 1 \) then \( T \) is a generalized star. If \( n(T') > 1 \), let \( v \) be a vertex of degree 1 in \( T' \) (such a vertex must exist since \( T' \) is a tree). Then \( v \) together with its pendent paths forms a pendent generalized star. □
Figure 4 A pendent generalized star

**Observation 3** The center $v$ of any pendent generalized star in tree $T$ (or the center of $T$ if $T$ itself is a nondegenerate generalized star) is central in $T$, since the center of any nondegenerate (pendent) generalized star has at least two pendent paths, so if $v$ were the end point of a path in a path cover there is at least one other pendent path of $v$ that could be adjoined to $v$ to reduce the number of paths in the path cover.

**Minimal Path Cover Algorithm** Let $T$ be a tree. Produce a list of paths in $T$ as follows:
1) Identify all pendent generalized stars in $T$ (or determine that itself $T$ is a generalized star)
2) If $T$ has one or more pendent generalized stars, for each pendent generalized star identified in (1),
   a. Identify the center $v$
   b. Add to the path list
      (i) the path consisting of $v$ and two pendent paths of $v$
      (ii) any remaining pendent paths of $v$
El s e $T$ itself is a generalized star
   If $T$ is degenerate, add it to the path list
   Else $T$ is nondegenerate
   a. Identify the center $v$
   b. Add to the path list
      (i) the path consisting of $v$ and two pendent paths of $v$
      (ii) any remaining pendent paths of $v$
3) Delete all the vertices of all the paths added to the path list in step (2) to obtain a new $T$
4) Repeat steps 1, 2 and 3 until out of vertices

One particular implementation of the Minimal Path Cover Algorithm is illustrated in Figure 5. The ovals indicate where the first round of pendent generalized stars were removed and the dashed curves indicate where the second round of pendent generalized stars were removed.
Figure 5 The Minimal Path Cover Algorithm applied to a tree

The path list produced by Minimal Path Cover Algorithm is clearly a path cover; such a path cover is called an external path cover. The name “external” is derived from the fact that the algorithm works from the outside in.

For a tree $T$ define the star number of $T$, denoted $\mathcal{S}(T)$, to be the total number of nondegenerate (pendent) generalized stars identified to produce an external path cover of $T$. For the tree $T$ in Figure 5, $\mathcal{S}(T) = 5$. Since the only differences between external path covers are which pendent paths of a particular (pendent) generalized star are chosen to be identified with the center, $\mathcal{S}(T)$ depends only on $T$ itself.

Let $S$ be a connected induced subgraph of tree $T$ such that $T - S$ is also connected. For a path cover $\mathbf{P}$ of $T$, let $\mathbf{P}|_S$ be the path cover of $S$ obtained by using the paths or parts of paths of $\mathbf{P}$ that lie in $S$. We say $\mathbf{P}$ covers $S$ if $\mathbf{P}|_S \subseteq \mathbf{P}$ and $\mathbf{P}|_{T - S} \subseteq \mathbf{P}$. In this case, $\mathbf{P}|_{T - S}$ is a minimal path cover of $T - S$. Note that for any pendent generalized star of $T$, any external path cover $\mathbf{P}$ covers $S$ and $T - S$, and $\mathbf{P}|_{T - S}$ is an external path cover of $T - S$. 


Theorem 4 An external path cover of tree $T$ is a minimal path cover of $T$.

Proof: We show by induction on $S(T)$ that $P$ is minimal. If $S(T) = 0$ then $T$ is a path; if $S(T) = 1$ then $T$ is a generalized star. In either of these cases an external path cover is clearly minimal.

Assume that for any tree $T$ such that $S(T) < k$, any external path cover is a minimal path cover. Let $S(T) = k > 1$, let $P$ be an external path cover of $T$, and let $S$ be a pendent generalized star. Let $v$ be the center of $S$, let $w$ be the anchor of $S$, let $P$ be the path of $P$ containing $v$, and let $P_i$, $i = 3, \ldots, r$, be the other paths (if any) of $P|_{S}$, i.e., other pendent paths of $v$. Denote the neighbor of $v$ in $P_i$ by $x_i$. Since $S(T - S) = k - 1$, by the induction hypothesis $P|_{T - S}$ is a minimal path cover. Construct a sequence of trees, $T_i$, and path covers, $P_i$, of $T_i$, as follows: $T_1 = T - S$ having path cover $P_1 = P|_{T - S}$, $T_2 = (T - S) + P$ having path cover $P_2 = P_1 \cup \{P\}$, $T_i = T_{i-1} + P_i$ having path cover $P_i = P_{i-1} \cup \{P_i\}$. Note $v$ is central $P$ and in each $T_i$ for $i > 1$ (the latter statement follows from Observation 3), so by Lemma 1, $P(T_i) = P(T_{i-1}) + 1$. Thus at each stage $P_i$ is a path cover with the right number of paths, so it must be minimal. Clearly $T_r = T$ and $P_r = P$. □

Note that not all minimal path covers are external. Figure 6 shows a minimal path tree whose associated minimal path cover is not external, and the minimal path tree of one external minimal path cover of the same tree.

![Figure 6](image1.png)  A minimal path tree that is not external

![Figure 6](image2.png)  An external path tree

We will show that any minimal path cover can be obtained from any other by a sequence of operations called trades, defined below. Let $P$ be a path cover of tree $T$. Recall that an edge of $T$ that does not appear in any of the paths of $P$ is called a connector edge (for $P$). The edges of $T$ which appear in paths of $P$ are called path edges for $P$. In many cases, whether a given edge is a connector edge or a path edge depends on the minimal path cover. In cases where it does not this will be very useful.

Choose a particular minimal path cover $P$ of $T$. The two end vertices of a connector edge for $P$ are called connector vertices (of $P$). Any high degree vertex must be a connector vertex of every path cover, but for low degree vertices, whether a vertex is a connector vertex may depend on the minimal path cover chosen. For a vertex $v$, any neighbors of $v$ that lie in the same path of $P$ are called path neighbors of $v$. Any neighbor of $v$ that is not a path neighbor is a connector neighbor of $v$. Any vertex has at most two path neighbors (in a particular path cover).

A connector vertex is called interior if it is an interior vertex of the path of $P$ in which it is contained; otherwise it is changeable. Notice that a vertex is changeable if and only if it is has at least one connector neighbor and at most one path neighbor. If a vertex is changeable in one minimal path cover, there is another minimal path cover in which it is not (see Lemma 17) so
changeability is not an intrinsic property of the vertex. In fact, we shall see that changeable vertices can be manipulated to get from one minimal path cover to another.

**Observation 5** A connector edge for a minimal path cover cannot have both of vertices be changeable, because if \{z,w\} is a connector edge with \(z\) a terminal vertex of path \(P_i\) and \(w\) a terminal vertex of path \(P_j\) then a path cover with fewer paths can be obtained by replacing \(P_i\) and \(P_j\) by the single path \(P_i + P_j\).

**Lemma 6 (Trading Lemma)** Let \(P\) be a minimal path cover of tree \(T\) which contains a changeable vertex \(z\). Let \{\(z,v\)\} be a connector edge for \(P\). See Figure 7. Then \(v\) must have 2 path neighbors \(x\) and \(y\), and there exist three distinct minimal path covers, \(P\), \(P_{x,z,y}\) and \(P_{y,z,x}\) of \(T\) with the following properties:

1) In \(P_{y,z,x}\), \(x\) is changeable and \(z\) and \(y\) are path neighbors of \(v\).
2) In \(P_{x,z,y}\), \(y\) is changeable and \(z\) and \(x\) are path neighbors of \(v\).
3) Each pair of \(P\), \(P_{x,z,y}\) and \(P_{y,z,x}\) agrees on every path but two.

Proof: Let the path \(P\) containing changeable vertex \(z\) be \((z, z_1, \ldots, z_r)\). Let the path \(Q\) containing \(v\) be \((x_r, \ldots, x_1, x, v, y, y_1, \ldots, y_s)\) (\(v\) must be interior because \(P\) is a minimal path cover, so \(x\) and \(y\) must exist). Define two new minimal path covers by using unchanged all the paths of \(P\) except \(P\) and \(Q\), and replacing \(P\) and \(Q\) by the two paths:

\((x_r, \ldots, x_1, x, v, z, z_1, \ldots, z_r)\) and \((y, y_1, \ldots, y_s)\) to obtain \(P_{x,z,y}\).

\((x_r, \ldots, x_1, x)\) and \((z_r, \ldots, z_1, z, v, y, y_1, \ldots, y_s)\) to obtain \(P_{y,z,x}\). □

Figure 7  The path trees associated with \(P\) and two different trades from \(P\)
Let \( z \) be a changeable vertex in minimal path cover \( P \) of \( T \). We say the minimal path cover \( P_{x,z:y} \) defined in the Trading Lemma is obtained by a trade from \( P \). After establishing a preliminary lemma, we will show that any minimal path cover can be obtained from any other by a sequence of trades.

**Lemma 7** Let \( S \) be a pendent generalized star of tree \( T \). For any minimal path cover \( P \) of \( T \), there is a minimal path cover \( P^*S \) of \( T \) such that \( P^*S \) covers \( S \), \( P^*S|_{T-S} = P|_{T-S} \), and \( P^*S \) is obtained from \( P \) by at most one trade.

Proof: Denote the center of \( S \) by \( v \), the pendent paths by \( P_i \) and the anchor of \( S \) by \( w \). If \( P \) covers \( S \), let \( P^*S = P \). If not, one pendent path, say \( P_1 \), is part of a path in \( P \) that includes \( v \), \( w \) and perhaps more vertices in \( T - S \). Let \( x \) be the neighbor of \( v \) in \( P_1 \) and \( y \) be the neighbor of \( v \) in another pendent path, \( P_2 \). Then let \( P^*S = P_{x,y:w} \) has the desired properties. \( \square \)

**Theorem 8 (Trading Theorem)** Let \( T \) be a tree. Any minimal path cover of \( T \) can be obtained from any other by a sequence of trades.

Proof: Note first that trades are reversible. We show that for \( Q \) any minimal path cover of tree \( T \) and \( P \) any external path cover of \( T \), then \( P \) is obtainable from \( Q \) in at most \( \mathcal{S}(T) \) trades. The proof is by induction on \( \mathcal{S}(T) \). If \( \mathcal{S}(T) = 0 \) then \( T \) is a path. If \( \mathcal{S}(T) = 1 \) then \( T \) is a generalized star, all minimal path covers are external and any such cover can be obtained from any other by one trade. Assume true whenever \( \mathcal{S}(T) < k \) and let \( \mathcal{S}(T) = k > 1 \). Identify one pendent generalized star of \( T \). Then \( Q^*S \) can be obtained from \( Q \) by at most one trade and we can choose \( Q^*S \) so that \( (Q^*S)|_S = P|_S \). Since \( \mathcal{S}(P|_{T-S}) = k - 1 \), \( P|_{T-S} \) is obtainable by a sequence of at most \( k-1 \) trades from \( (Q^*S)|_{T-S} = Q|_{T-S} \). \( \square \)

Although changeability depends on a particular path tree, the property of being changeable for some path cover is an intrinsic property of the vertex \( v \) (in tree \( T \)). We say a vertex \( v \) of tree \( T \) is potentially changeable if there exists a minimal path cover \( P \) such that \( v \) is changeable in \( P \). We will show that the next algorithm identifies exactly the potentially changeable vertices of \( T \).

**Marking Algorithm** Let \( P \) be a path cover of \( T \).

0) Mark every vertex of \( T \) that is a changeable vertex in \( P \). Such a vertex is called 0-step-changeable.

1) Mark every unmarked vertex that is a path neighbor of a connector neighbor of a \( k \)-step-changeable vertex. Such a vertex is called \( k+1 \)-step-changeable.

2) Repeat step (1) until done.

See Figure 8.
If $w_k$ is marked as a $k$-step-changeable vertex, there are vertices $v_i$ and $w_{i-1}, i=1,\ldots,k$ such that, in $\mathbf{P}$, $w_0$ is changeable, $\{w_{i-1}, v_i\}$ is a connector edge, and $w_i$ is a path neighbor of $v_i$. Call the path $(w_0, v_1, w_1, \ldots, v_k, w_k)$ a changeability path of $w_k$ (it is a path in $T$ but is not one of the paths in $\mathbf{P}$).

**Lemma 9** Apply the Marking Algorithm to tree $T$ using one minimal path cover $\mathbf{P}$.

1) If $v$ is marked as $k$-step-changeable then it is possible to go from $\mathbf{P}$ to a minimal path cover $\mathbf{Q}$ in which $v$ is changeable by a sequence of $k$ trades.
2) If $\{z,v\}$ is a connector edge for $\mathbf{P}$ with changeable vertex $z$ and path neighbors $x$ and $y$ of $v$, then a vertex $u$ of $T$ is marked using $\mathbf{P}$ and if and only if $u$ is marked using $\mathbf{P}_{x,z:y}$

Proof: Let $w_k$ be marked as a $k$-step-changeable vertex and let $(w_0, v_1, w_1, \ldots, v_k, w_k)$ be a changeability path of $w_k$. Notice that the $w_i$ are all in distinct paths, since there are no cycles in $T$. Let the other path neighbor of $v_i$ be denoted $x_i$ and let $\mathbf{P}_0 = \mathbf{P}$. The result (1) follows by repeated application of the Trading Lemma to produce path covers $\mathbf{P}_{i+1} = (\mathbf{P}_i)_{x_{i+1}, w_i; w_{i+1}}$ having $w_{i+1}$ changeable.

To establish (2), suppose $u$ is marked using $\mathbf{P}$. If $z$ starts the changeability path of $u$, replace $z$ by $y, v, z$, to obtain a changeability path for $\mathbf{P}_{x,z:y}$. Trades are reversible, so the converse follows. □

**Theorem 10 (Marking Theorem)** A vertex $z$ of tree $T$ is potentially changeable if and only if it is marked by the Marking Algorithm applied to $T$ using any one minimal path cover $\mathbf{P}$.

Proof: Any marked vertex is potentially changeable by (1) of Lemma 9.

If vertex $z$ is potentially changeable, then there exists a minimal path cover $\mathbf{Q}$ of $T$ in which $z$ is changeable, so $z$ is marked by the Marking Algorithm using $\mathbf{Q}$. By the Trading Theorem, $\mathbf{P}$ can be obtained from $\mathbf{Q}$ by trading. By Lemma 9 (2), the markings of $T$ do not change when trading, so $z$ is marked by applying the Marking Algorithm using $\mathbf{P}$. □

The concept of potential changeability, or rather lack thereof, can be exploited to determine features that must be common to all minimal path covers and determine when a minimal path cover is unique.

**Theorem 11** Let $\{v,w\}$ be a connector edge for one minimal path cover of tree $T$. Then $v$ and $w$ cannot both be potentially changeable. One of $v$ or $w$ is potentially changeable if and only if $\{v,w\}$ is a path edge for some other minimal cover of $T$. 

![Figure 8 The result of the Marking Algorithm showing $k$-step changeability]
Proof: Suppose both $v$ and $w$ are potentially changeable. Choose a minimal path cover $P$ of $T$. By Theorem 10, both $v$ and $w$ are marked by the Marking algorithm using $P$. Let $w = w_k$ and $v = v_m$ have changeability paths $(w_0, z_1, w_1, ..., z_k, w_k)$ and $(v_0, u_1, v_1, ..., u_m, v_m)$ respectively, with $w_i$ in path $P_i$ and $v_i$ in path $Q_i$ of $P$. Note first that the changeability paths cannot intersect, because if they intersected at vertex $x$, then the edge $\{v, w\}$, the part of the changeability path of $v$ between $v$ and $x$, and the part of the changeability path of $w$ between $w$ and $x$ would contain one or more cycles. Thus $z_i, w_i, u_i,$ and $v_i$ must all be distinct. Then the sets $\{P_i: i=1, \ldots, k\}$ and $\{Q_i: i=1, \ldots, m\}$ are disjoint, because if some $P_i = Q_j$ this would again create a cycle. Thus by repeated application of the Trading Lemma we can create a minimal path cover with a connector having both ends changeable, contradicting Observation 5. So it is not possible to have both ends of a connector potentially changeable.

If $v$ is potentially changeable, there is a path cover in which $v$ is changeable. If $\{v, w\}$ is not already a path edge, apply the Trading Lemma to produce a minimal path tree in which $\{v, w\}$ is a path edge.

For the converse, let $\{v, w\}$ be a connector edge for minimal path cover $P$ of $T$ and let $\{v, w\}$ be a path edge for minimal path cover $Q$ of $T$. We show that one of $v$ and $w$ is potentially changeable. If $v$ is changeable, we are done. If not, $\deg_T v \geq 3$. Define a sequence of vertices $v_i$ and $w_i$ with $\{v_i, w_i\}$ a connector edge for $P$ and a path edge for $Q$, and $\{w_i, v_{i+1}\}$ a path edge for $P$ and a connector edge for $Q$ as follows (see Figure 9): $v_0 = v$, $w_0 = w$. Assume $v_i$ and $w_i$ have been defined. If $w_i$ is not changeable for $P$, it has two path neighbors $x, y$ in $P$ (and $\deg_T w_i \geq 3$). Since one of the path neighbors of $w_i$ in $Q$ is $v_i$, at least one of $x, y$ is a connector neighbor of $w_i$ in $Q$. Rename such a neighbor $v_{i+1}$. If $v_{i+1}$ is not changeable in $Q$, it has two path neighbors $u, z$ in $Q$ (and $\deg_T v_{i+1} \geq 3$). Since one of the path neighbors of $v_{i+1}$ in $P$ is $w_i$, at least one of $u, z$ is a connector neighbor of $v_{i+1}$ in $P$. Rename such a neighbor $w_{i+1}$.
All $v_i$, $w_i$ are distinct since $T$ does not have any cycles. Thus this produces a path $(v_0, w_0, v_1, w_1, \ldots, v_k, w_k, \ldots)$ in $H$, the subgraph induced by the high degree vertices of $T$. No path in $H$ can contain more vertices than the diameter of this component of $H$ (the diameter of a tree $T$ is the maximum number of vertices in a path in $T$). The only way for this path to terminate is for it to produce a changeable vertex, $v_{i+1}$ in $Q$ or $w_{i+1}$ in $P$. If $v_{i+1}$ is changeable in $Q$ (respectively, $w_{i+1}$ is changeable in $P$) then the Marking Algorithm using $Q$ (respectively, $P$) marks $v$ (respectively, $w$), so $v$ (respectively, $w$) is potentially changeable.

**Corollary 12** If edge $\{v, w\}$ is an edge of tree $T$ and both $v$ and $w$ are potentially changeable then $\{v, w\}$ is always a path edge. If neither $v$ and $w$ is potentially changeable then $\{v, w\}$ is always a path edge or always a connector edge.

Edges with both vertices potentially changeable, or neither vertex potentially changeable are called *forced*, because such an edge is forced to assume the same role in every minimal path tree. Forced connector edges are especially useful, because they can be used to break apart minimal path trees into smaller pieces for analysis. Let $P$ be a minimal path cover of tree $T$. The forest obtained from $T$ by deleting all forced connector edges for $P$ is called the *dissection* $T$.

**Observation 13** The dissection of $T$ is independent of the path cover used to create it, because every edge removed was a forced connector edge, and so is a connector edge for any minimal path cover.
We can use the dissection of a tree to reduce the problem of counting of the number of minimal path trees of $T$ to the problem of counting the number in each component of the dissection of $T$. Let $\#\text{MPC}(T)$ denote the number of minimal path covers of $T$.

**Theorem 14** Let $T_1, T_2, \ldots, T_k$ be the components of the dissection of $T$.

1) Every minimal path cover of $T$ is obtained as a union of disjoint minimal path covers of $T_1, T_2, \ldots, T_k$, and any such union is a minimal path cover of $T$.

2) $\#\text{MPC}(T) = \prod_{i=1}^{k} \#\text{MPC}(T_i)$

3) A vertex of $T_i$ is changeable in $P|_{T_i}$ if and only if it is changeable in $P$.

4) A vertex of $T_i$ is potentially changeable in $T_i$ if and only if it is potentially changeable in $T$.

Proof: The first statement is immediate from the fact that any minimal path cover of $T$ covers all $T_1, T_2, \ldots, T_k$, because every edge removed in the dissection was a forced connector edge, and so is a connector edge for any minimal path cover. The second follows from the first. For (3), let $v$ be changeable in a minimal path cover of $T_i$. Taking the union of this minimal path cover with minimal path covers of the other components results in a minimal path cover $P$ for $T$. Since $P$ is obtained from the union of the minimal path trees by joining them with additional connector edges, $v$ is changeable in $P$. For the converse, let vertex $v$ of $T_i$ be changeable in minimal path cover $P_i$ of $T$. Any connector edge for $P$ that is incident with $v$ has a potentially changeable vertex (namely $v$) and so is not forced and is not removed in the dissection. Since no connector incident with $v$ is removed in the dissection, $v$ is still a connector vertex in $P|_{T_i}$, and obviously has at most one path neighbor, so is changeable in $P|_{T_i}$. Finally, (4) follows from (3) from the definition of potentially changeable. □
A path cover $P$ of $T$ is interior if every connector vertex in $P$ is interior. It follows from the Trading Lemma that a minimal path cover is not unique if it is not interior. Once we show an interior path cover is minimal, it follows from the dissection theorem that it is the unique minimal path cover of $T$.

The path $P_j$ is final if $P_j$ has exactly one connector edge incident with its vertices. Any path cover has at least two final paths, just as any tree has at least two vertices of degree one.

**Theorem 15** A path cover with at most one changeable vertex is a minimal path tree.

**Proof:** By induction on $|P|$. The result is clear when $|P| = 1$. Assume any path cover $Q$ with at most one changeable vertex such that $|Q| < k$ is a minimal path tree. Let $|P| = k > 1$. There must be at least 2 final paths in $P$; choose a final path $P$ for which the (only) connector vertex $v$ of $P$ is an interior vertex of $P$. Let $u$ be the other vertex of this connector edge. Note $T - P$ is a tree and $P$ covers $T - P$. $|P|_{T - P} = k - 1$. $T = P + (T - P)$. Note $P|_{T - P}$ is a minimal path cover for $T - P$. Since $v$ is an interior vertex of $P$, it is central in $P$, so $P(T) = P(P + (T - P)) = P(P) + P(T - P) = 1 + (k-1) = k = |P|$, and so $P$ is a minimal path cover of $T$. □

**Corollary 16** A minimal path cover $P$ of $T$ is the unique minimal path cover of $T$ if and only if it is an interior path cover.

**Proof:** Since an interior vertex is not changeable, and an interior path cover is a minimal path cover, no vertices are marked by the Marking Algorithm using the interior path cover, so no vertices are potentially changeable. Therefore every connector edge in the interior path cover is a forced connector edge, and the dissection of $T$ is the disjoint union of the paths in the given interior path cover. Since a minimal path cover that is not interior has a changeable vertex, the converse follows immediately from the Trading Lemma.

Thus a tree having an interior path cover can be referred to as an *interior tree*, since it has only this one minimal path tree.

A changeable vertex $z$ in $P$ is called doubly changeable if there are at least 2 connector edges incident with $z$ and $z$ has no path neighbors; otherwise $z$ is called singly changeable (in this case either $z$ has one path neighbor or $\deg_T z = 1$). We will ultimately be able to describe and count the minimal path trees of a tree that has a minimal path tree with one singly changeable vertex. But first we need examine the effect of trades on single and double changeability.

**Lemma 17** Let $P$ be a minimal path cover of tree $T$. Let $z$ be a changeable vertex with connector edge $\{z,v\}$, with $v$ having path neighbors $x$ and $y$.

1) If $z$ is singly changeable in $P$, then $z$ is not changeable in $P_{x,z,y}$.

2) If $z$ is doubly changeable in $P$, then $z$ is singly changeable in $P_{x,z,y}$; $z$ has at least one other connector neighbor $w$, $w$ has two path neighbors $u,t$, and $z$ is not changeable in $(P_{x,z,y})_{t,z,w}$.

3) If $y$ is not changeable in $P$, then $y$ is singly changeable in $P_{x,z,y}$.

4) If $y$ is singly changeable in $P$, then $y$ is doubly changeable in $P_{x,z,y}$.

5) It is not possible for $y$ to be doubly changeable in $P$. 

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Proof: If $z$ is singly changeable in $P$, then either $z$ has a path neighbor in $P$ or $\deg_T z = 1$. If $z$ has a path neighbor $z_1$ then $z$ is not changeable in $P_{x,z,y}$ because it has the two path neighbors $z_1$ and $v$ in $P_{x,z,y}$ (see Figure 7, which illustrates the Trading Lemma). If $\deg_T z = 1$ then $z$ is not changeable in $P_{x,z,y}$ because it has no connectors in $P_{x,z,y}$.

If $z$ is doubly changeable in $P$, then $z$ has at least one other connector neighbor $w$ besides $v$ in $P$, so $z$ has at least one connector neighbor in $P_{x,z,y}$, and only one path neighbor ($v$), so $z$ is singly changeable in $P_{x,z,y}$. Since $w$ cannot be changeable (or the connector edge $\{z, w\}$ would have both vertices changeable) $w$ has two path neighbors $u$, $t$, and $z$ is not changeable in $(P_{x,z,y})_{t,z,u}$ because $z$ now has the two path neighbors $v$ and $w$. See Figure 11.

If $y$ is not changeable in $P$, then either $y$ does not have a connector neighbor or $y$ has two path neighbors. If $y$ does not have a connector neighbor in $P$, then $y$ has only one connector neighbor ($v$) in $P_{x,z,y}$, so $y$ cannot be doubly changeable. If $y$ has two path neighbors in $P$, then $y$ has one path neighbor in $P_{x,z,y}$, so $y$ cannot be doubly changeable. In either case, since $y$ is changeable and not doubly changeable in $P_{x,z,y}$, it must be singly changeable in $P_{x,z,y}$.

If $y$ is singly changeable in $P$, then $y$ has a connector neighbor $w$ and $y$ has exactly one path neighbor $v$ in $P$ (see Figure 12). So $y$ has at least two connector neighbors ($v$ and $w$) and no path neighbors in $P_{x,z,y}$, so $y$ is doubly changeable.

It is not possible for $y$ to be doubly changeable in $P$, since it has at least one path neighbor, $v$. □

Figure 11 Two trades to make a doubly changeable vertex not changeable.
As noted earlier, parts (1) and (2) of Lemma 17 show that if \( z \) is changeable in one minimal path tree of \( T \), there is another minimal path cover of \( T \) in which \( z \) is not changeable.

Let \( P \) be a minimal path cover of tree \( T \). Define the simultaneous changeability of \( T \) to be the number of singly changeable vertices in \( P \) plus twice the number of doubly changeable vertices in \( P \). As defined, the simultaneous changeability of \( T \) appears to depend on the minimal path tree used to compute it, but this in not the case, as the next theorem shows.

**Theorem 18** The simultaneous changeability of \( T \) is independent of the minimal path tree chosen to evaluate it.

Proof: Any minimal path cover is obtainable from any other by a sequence of trades, and by Lemma 17, the trade from \( P \) to \( P_{x,z,y} \) reduces the changeability of \( z \) by one and increases the changeability of \( y \) by one, so the simultaneous changeability remains constant from \( P \) to \( P_{x,z,y} \). Thus trading does not affect the simultaneous changeability. □

A path cover \( P \) of \( T \) is *almost interior* if exactly one connector vertex of \( P \) is changeable, and this vertex is singly changeable. It was shown in Theorem 15 that any almost interior path tree is a minimal path tree. Again, the tree \( T \) itself can also be referred to as almost interior, since the simultaneous changeability of such a tree \( T \) is one, so every minimal path tree of \( T \) is almost interior.

**Theorem 19** If tree \( T \) is an almost interior tree, the number of minimal path covers trees of \( T \) is the number of potentially changeable vertices of \( T \).

Proof: Let \( T \) be almost interior tree. We show that there is a one-to-one correspondence between the potentially changeable vertices of \( T \) and the minimal path covers of \( T \). Each minimal path cover has exactly one changeable vertex, so the map from minimal path cover to potentially changeable vertices of \( T \) is a well-defined function. Each potentially changeable vertex is changeable in some minimal path cover so this map is surjective.

The proof that the one changeable vertex of a minimal path tree uniquely determines the minimal path cover is by induction on the star number of \( T \), \( \mathcal{S}(T) \). If \( \mathcal{S}(T) = 0 \), then \( T \) is a path. If \( \mathcal{S}(T) = 1 \), then \( T \) is a generalized star with 3 paths pendent from the center. Choosing the changeable vertex uniquely determines the minimal path cover.

Assume that for any almost interior tree \( T \) such that \( \mathcal{S}(T) < k \), a minimal path cover is uniquely determined by its changeable vertex. Let \( \mathcal{S}(T) = k > 1 \), let \( P \) be an minimal path cover of
In which vertex \( z \) is changeable. Since \( T \) has at least two pendent generalized stars, let \( S \) be a pendent generalized star of \( T \) that does not contain \( z \). Denote the center of \( S \) by \( v \). Then \( v \) has exactly two pendent paths (since \( S \) has no changeable vertices in \( P \)) and in \( P \) the two pendent paths of \( v \) are joined to \( v \) to form one path. Furthermore, \( P_{T-S} \) is an almost interior path tree with changeable vertex \( z \), and so \( T - S \) is an almost interior tree. In any minimal path cover \( Q \) of \( T \) in which vertex \( z \) is changeable, no vertex of \( S \) is changeable, so the two pendent paths of \( v \) must again be joined to \( v \) to form a single path. Thus \( Q \) covers \( S \) and \( T - S, Q_{T-S} \) is a almost interior path tree of \( T - S \) with changeable vertex \( z \). Since \( S(T - S) = k - 1 \), by the induction hypothesis, \( Q_{T-S} = P_{T-S} \), and \( Q = P \). ☐

**Example 2** Figure 13 shows the tree of Figure 10, drawn as a minimal path tree. The potentially changeable vertices are shaded, and the dashed curves indicate the components \( T_1, T_2, T_3 \) and \( T_4 \) of the dissection of \( T \). Since each component is interior or almost interior, by Corollary 16 and Theorem 19, we have: \#MPC(\( T_1 \)) = 3, \#MPC(\( T_2 \)) = 1, \#MPC(\( T_3 \)) = 5, \#MPC(\( T_4 \)) = 3. Thus by Theorem 14, \#MPC(\( T \)) = 45.

![Figure 13](image)

In [2], an additional parameter of a tree \( T, \Delta(T) = \max\{p - q: \text{the deletion of a set of } q \text{ vertices from } T \text{ leaves } p \text{ paths}\} \) was defined and it was shown that \( \Delta(T) = P(T) \) (= the maximum multiplicity of an eigenvalue in a symmetric matrix whose graph is \( T \)). The Minimal Path Cover Algorithm identifies a set of vertices whose deletion realizes \( \Delta(T) \).

**Proposition 20** In the Minimal Path Cover Algorithm, the set of vertices identified as centers of all nondegenerate (pendent) generalized stars form a set of \( S(T) \) vertices whose deletion leaves \( P(T) + S(T) \) paths, and thus realizes \( \Delta(T) \).

Proof: When a nondegenerate (pendent) generalized star with \( r \) pendent paths having center \( v \) is identified, \( r - 1 \) paths are added to the path cover and \( r \) paths are created by the deletion of the one vertex \( v \). ☐

Note that Johnson and Saiago [3] provided the following algorithm for producing a set of \( q \) vertices to realize \( \Delta(T) \):
**Johnson-Saiago \( \Delta \) Algorithm**

1. Delete all high degree vertices \( v \) such that \( \deg_H v \leq \deg_T v - 2 \).
2. Repeat until no high degree vertices remain.

Both algorithms proceed from the outside in but can yield slightly different sets as illustrated in the next example.

**Example 3** The tree \( T \) shown in Figure 14 has 3 centers of nondegenerate stars identified by the Minimal Path Algorithm (black in the left picture) and the deletion of these 3 vertices leaves 7 paths. The Johnson-Saiago \( \Delta \) Algorithm applied to \( T \) deletes the 4 vertices colored black in the right picture, leaving 8 paths. Note that neither algorithm produced the minimal set realizing \( \Delta(T) \), namely the single vertex in the center of the drawing.

![Figure 14 Two different \( \Delta \) algorithms applied to \( T \)](image)

**References**

[1] F. Barioli, S. Fallat, and L. Hogben, Computation of minimal rank and path cover number of certain graphs, pre-print.
