1. Extremal graph theory
2. Edit Distance Origins
3. Definitions related to Edit Distance
4. Hereditary Properties
5. Binary Chromatic Number
6. The Edit Distance Function
7. Weighted Turán lemma
8. Future Work
9. Thanks!
A classical extremal problem

Suppose we have an \( n \)-vertex graph \( G \).
A classical extremal problem

Suppose we have an $n$-vertex graph $G$.

We want to compute the fewest number of edge-additions plus edge-deletions to apply to $G$ so that the resulting graph $H$ has no triangles.

Theorem (Mantel, 1907)
If an $n$-vertex graph $H$ has no triangles, then the number of edges $H$ has is at most $\left\lfloor \frac{n^2}{4} \right\rfloor$. This bound is only achieved if $H$ is complete bipartite.
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So, if $G$ were the complete graph, it would require at least

$$\binom{n}{2} - \lfloor n^2/4 \rfloor$$

edge-deletions.
Editing away triangles

So, if $G$ were the complete graph, it would require at least

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\binom{n}{2} - \lfloor n^2/4 \rfloor = \left( \left\lfloor \frac{n}{2} \right\rfloor \right)^2 + \left( \left\lceil \frac{n}{2} \right\rceil \right)^2
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But for any graph $G$, we can delete at most this many edges and remain triangle-free:

Partition the vertices in half and delete edges inside each part.
Results on triangles

So, the maximum number of changes required to remove triangles from $n$-vertex graph $G$ is

$$\left(\left\lfloor \frac{n}{2} \right\rfloor \right)^2 + \left(\left\lceil \frac{n}{2} \right\rceil \right)^2.$$ 

This is achieved by $G = K_n$. 

Theorem (Turán, 1941)

If an $n$-vertex graph $H$ has no copy of $K_{\ell + 1}$, then the number of edges $H$ has is at most $\frac{n^2}{2\ell}$. This bound is only achieved if $H$ is complete $\ell$-partite and $\ell | n$. 

So, the maximum number of changes required to remove triangles from $n$-vertex graph $G$ is

$$\sim \frac{n^2}{2^\ell} \sim \frac{1}{\ell^2} \left(\frac{n^2}{2}\right).$$ 

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Ryan Martin (Iowa State U.)
Results on triangles

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The edit distance question began with the following question of Chen, Eulenstein, Fernández-Baca and Sanderson:
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**Question**

How many edge-deletions plus edge-additions are necessary to ensure that \( G \) has no copy of “\( W \)” as an induced subgraph?

An induced “\( W \)” (edges in red, nonedges in blue):

```
A  ●  ●  ●  ●  ●  ●  ●  ●
  W
B  ●  ●  ●  ●  ●  ●  ●  ●
```
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The question relates to consensus trees. Two trees are comparable if a corresponding bipartite graph has no induced $W$ or $M$. 
Szemerédi’s Regularity Lemma gives:

**Theorem (Axenovich-M. 2006)**

Let $H$ be any fixed bipartite graph that is neither empty nor complete. The number of edge-operations necessary to remove all induced copies of $H$ from a random $N \times N$ bipartite graph is at least $(1/2)N^2 - o(N^2)$, with high probability.

Clearly, $(1/2)N^2$ edge operations suffices.
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Clearly, $(1/2)N^2$ edge operations suffices.

The more general case is not so easy to settle, even asymptotically.
Given: Labeled graphs $G$ and $G'$, each on $n$ vertices.
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Definition

The **EDIT DISTANCE BETWEEN $G$ AND $G'$**

$$\text{Dist}(G, G') = |E(G) \triangle E(G')|$$

is the size of the symmetric difference of the edge sets.
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That is, it is the minimum number of edge-additions plus edge-deletions to transform $G$ into $G'$.
Given: A labeled graph \( G \) and a graph property \( \mathcal{P} \).
A **GRAPH PROPERTY** is a set of graphs.

The edit distance from \( G \) to \( \mathcal{P} \)

\[
\text{Dist}(G, \mathcal{P}) = \min \{ \text{Dist}(G, G') : G' \in \mathcal{P} \}
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is the least edit distance of \( G \) to a graph in \( \mathcal{P} \).
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Edit Distance from properties

**Given:** A labeled graph $G$ and a graph property $\mathcal{P}$.
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The **EDIT DISTANCE FROM** $G$ **TO** $\mathcal{P}$

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The **EDIT DISTANCE FROM $\mathcal{P}$**

$$\text{Dist}(n, \mathcal{P}) = \max \{ \text{Dist}(G, \mathcal{P}) : |V(G)| = n \}$$

is the maximum edit distance of an $n$-vertex graph to a graph in $\mathcal{P}$. 
Extremal Edit Distance

**Given:** A natural number $n$ and a graph property $\mathcal{P}$.

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**Extremal Edit Distance**

**Given:** A natural number \( n \) and a hereditary graph property \( \mathcal{H} \).

**Definition**

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A **hereditary property** is one that is preserved under vertex-deletion. Example: $\text{Forb}(K_{3,3})$, no induced copy of $K_{3,3}$. 
Hereditary Properties

Definition

A **HEREDITARY PROPERTY**, \( \mathcal{H} \), of graphs is one that holds under the deletion of vertices.

Examples:
- Planar graphs.
- Line graphs of bipartite graphs.
- Chordal graphs: graphs with no chordless cycle of length \( \geq 4 \).
- Perfect graphs: \( \chi = \omega \) for all induced subgraphs.
- \( \text{Forb}(H) \): graphs with no induced copy of \( H \).

\( \text{Forb}(H) \) is a principal hereditary property.

For the rest of this talk, all of our hereditary properties are principal; i.e., \( H = \text{Forb}(H) \), for some graph \( H \).
A **hereditary property**, \( \mathcal{H} \), of graphs is one that holds under the deletion of vertices.

I.e., if \( G \in \mathcal{H} \), then every induced subgraph of \( G \) is in \( \mathcal{H} \).

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A 5-cycle as a subgraph, but no induced 5-cycle.
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\[\mathcal{H} = \text{Forb}(H), \text{ for some graph } H.\]
A useful parameter

Let $\mathcal{H} = \text{Forb}(H)$ and let $a, c \in \mathbb{N}$ have the property that $V(H)$ cannot be partitioned into a set of $a$ independent sets and $c$ cliques.

But $V(H)$ can be partitioned into ANY combination of $a + c + 1$ independent sets and cliques.
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We call $a + c + 1$ THE BINARY CHROMATIC NUMBER, $\chi_B(\mathcal{H})$. 
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Theorem (Axenovich-Kézdy-M., 2008)

Let $\mathcal{H} = \text{Forb}(H)$ for some fixed graph $H$ which has binary chromatic number $\chi_B(H)$,

$$\text{Dist}(n, \text{Forb}(H)) \geq \frac{1}{2(\chi_B(H) - 1)} \binom{n}{2} - o(n^2).$$
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$$\text{Dist}(n, \text{Forb}(H)) \geq \frac{1}{2(\chi_B(H) - 1)} \binom{n}{2} - o(n^2).$$

Moreover, this holds with equality if $H$ is self-complementary.
An example: 5-cycle

We will compute $\chi_B$ for the 5-cycle, $C_5$. 

So, $\chi_B(C_5)^3$.

But it cannot be partitioned into, say, 2 independent sets and 0 cliques.

So, $\chi_B(C_5)^3$. 

Ryan Martin (Iowa State U.)

The edit distance in graphs

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An example: 5-cycle

We will compute $\chi_B$ for the 5-cycle, $C_5$.

Let us consider all $(a, c)$ such that $a + c + 1 = 3$:  

The 5-cycle can be partitioned into 3 independent sets and 0 cliques.
An example: 5-cycle

We will compute $\chi_B$ for the 5-cycle, $C_5$.

Let us consider all $(a, c)$ such that $a + c + 1 = 3$:

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So, $\chi_B(C_5) \leq 3$. 

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So, $\chi_B(C_5) \geq 3$. 
An example: 5-cycle

We will compute $\chi_B$ for the 5-cycle, $C_5$.

So, $\chi_B(C_5) = 3$.

But it cannot be partitioned into, say, 2 independent sets and 0 cliques.

So, $\chi_B(C_5) = 3$. 
Since $\chi_B(C_5) = 3$, and $C_5$ is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(H)) = \frac{1}{2(\chi_B(H) - 1)} \binom{n}{2} - o(n^2).$$
Since $\chi_B(C_5) = 3$, and $C_5$ is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(H)) = \frac{1}{2(3 - 1)} \binom{n}{2} - o(n^2).$$
Edit distance for \( C_5 \)

Since \( \chi_B(C_5) = 3 \), and \( C_5 \) is self-complementary, the theorem gives

\[
\text{Dist}(n, \text{Forb}(H)) = \frac{1}{4} \binom{n}{2} - o(n^2).
\]
Since $\chi_B(C_5) = 3$, and $C_5$ is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(H)) = \frac{1}{4} \binom{n}{2} - o(n^2).$$

Normalize and take the limit:

$$\text{Dist}(n, \text{Forb}(H)) \left( \frac{n}{2} \right)^{-1} = \frac{1}{4} - o(1).$$
Since $\chi_B(C_5) = 3$, and $C_5$ is self-complementary, the theorem gives

$$\text{Dist}(n, \text{Forb}(H)) = \frac{1}{4} \binom{n}{2} - o(n^2).$$

Normalize and take the limit:

$$\lim_{n \to \infty} \text{Dist}(n, \text{Forb}(H)) \left( \frac{n}{2} \right)^{-1} = \frac{1}{4}.$$  

We denote

$$d^*(\mathcal{H}) \overset{\text{def}}{=} \lim_{n \to \infty} \text{Dist}(n, \mathcal{H}) \left( \frac{n}{2} \right)^{-1}.$$
Let $G_{n,p}$ denote the Erdős-Rényi random graph:

I.e., there are $n$ vertices and each edge is present, independently, with probability $p$. 
Understanding $d^*$

Let $G_{n,p}$ denote the Erdős-Rényi random graph:

i.e., there are $n$ vertices and each edge is present, independently, with probability $p$.

**Theorem (Alon-Stav, 2008)**

For every hereditary property, $\mathcal{H}$, there exists a $p^* = p^*(\mathcal{H}) \in [0, 1]$ such that

$$d^*(\mathcal{H}) = \lim_{n \to \infty} \text{Dist} \left( G_{n,p^*}, \mathcal{H} \right) \left( \binom{n}{2} \right)^{-1}.$$  

(The expression $\text{Dist} \left( G_{n,p^*}, \mathcal{H} \right)$ is tightly concentrated about the mean, so the right-hand side is well-defined.)
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Theorem (Balogh-M., 2008)

For every hereditary property, $\mathcal{H}$, and every $p \in [0, 1]$, if

$$g_{\mathcal{H}}(p) = \lim_{n \to \infty} \text{Dist} \left( G_n, p, \mathcal{H} \right) \left( \binom{n}{2} \right)^{-1}$$

then it is also true that

$$g_{\mathcal{H}}(p) = \lim_{n \to \infty} \max \left\{ \text{Dist} \left( G, \mathcal{H} \right) : |V(G)| = n, |E(G)| = p \left( \binom{n}{2} \right) \right\} \left( \binom{n}{2} \right)^{-1}.$$
Theorem (Balogh-M., 2008)

For every hereditary property, $\mathcal{H}$, and every $p \in [0, 1]$, if

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Roughly, the hardest density-$p$ graph to edit is the random graph.
Properties of $g_{\text{Forb}(H)}(p)$

- Continuous and concave down.
- Achieves its maximum ($p^*, d^*$) for some $p^* \in [0, 1]$.
- $g_{H}(p) = \frac{1}{2} (\chi_B(H) - 1)$.
- If $H$ is neither complete nor empty, then $g_H(0) = g_H(1) = 0$.
- For any rational $r \in [0, 1]$, there is an $H$, such that $p^*(\text{Forb}(H)) = r$.
- There is an irrational $q \in [0, 1]$ and an $H$, such that $p^*(\text{Forb}(H)) = q$.

Theorem (Balogh-M., 2008)

- $p^*(\text{Forb}(K_3, K_3)) = \sqrt{2} - 1$.
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**Theorem (Balogh-M., 2008)**

\[
p^*_{\text{Forb}(K_3,K_3)} = \sqrt{2} - 1 \quad \text{and} \quad d^*_{\text{Forb}(K_3,K_3)} = 3 - 2\sqrt{2}.
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The Edit Distance Function

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**Theorem (Balogh-M., 2008)**

$$p^*(\text{Forb}(K_{3,3})) = \sqrt{2} - 1 \quad d^*(\text{Forb}(K_{3,3})) = 3 - 2\sqrt{2}.$$
Weighted Turán lemma

The value of computing the edit distance for specific hereditary properties are the techniques used and how they might be generalized.

Lemma (Balogh-M., 2008)

Let $a \geq 2$ and $K, |V(K)| = k$ be a graph with edges colored BLACK, WHITE and GRAY, with the property that any set $A$ of $a$ vertices has at least one of the following conditions:

1. $A$ contains at least one WHITE edge;
2. $A$ contains a spanning subgraph of BLACK edges.

With $E_W(K)$ denoting the white edges and $E_B(K)$ the black edges, $(a-1)|E_W(K)| + |E_B(K)| \geq \lceil k^2 (k-a+1) \rceil$.

If we only apply (1) and not (2), then, with $a$ fixed, Turán’s theorem is, asymptotically, a consequence.
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With \( EW(K) \) denoting the white edges and \( EB(K) \) the black edges,

\[
(a - 1)|EW(K)| + |EB(K)| \geq \left\lceil \frac{k}{2}(k - a + 1) \right\rceil.
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Future Work

- $g_\mathcal{H}(p)$ is a function of $\mathcal{H}$ and is derived from the random graphs $G_{n,p}$. What other functions of $\mathcal{H}$ have similar properties? The following function has been studied and shares some of the properties of $g_\mathcal{H}(p)$:
  \[- \log_2 \Pr(G_{n,p} \in \mathcal{H}).\]


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- For which metrics on graph spaces do similar properties hold to the (normalized) edit metric? The metric derived from the so-called cut norm, for example, is well-studied and shares some properties with edit.
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What are the axioms that will define such metrics as being “good”?
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- Graph limits are new but well-studied. The limits of graphs are functions of a measure space on $[0, 1]^2$. Balázs Szegedy reports that limits of hereditary properties can be identified. Graph limits and graph metrics may share a common theoretical bond.

Note that this would imply the counterintuitive $p^* \sim \log(1-p^0) \log(p^0(1-p^0))$. Counterintuitive because $\log(1-p^0) \log(p^0(1-p^0)) = p^0$ if and only if $p^0 \in \{0, 1/2, 1\}$.
Future Work

Conjecture

Fix $p_0 \in [0, 1]$ and let $H \sim G(n_0, p_0)$ with $\mathcal{H} = \text{Forb}(H)$. Then,

$$g_{\mathcal{H}}(p) \sim \frac{2 \log_2 n_0}{n_0} \min \left\{ \frac{p}{\log_2 \frac{1}{1-p_0}}, \frac{1-p}{\log_2 \frac{1}{p_0}} \right\}.$$

with prob. $\rightarrow 1$ as $n_0 \rightarrow \infty$. Note that this would imply the counterintuitive $p^* \sim \log(1-p_0) \log(p_0(1-p_0))$. Counterintuitive because $\log(1-p_0) \log(p_0(1-p_0)) = p_0$ if and only if $p_0 \in \{0, \frac{1}{2}, 1\}$. 

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