Math 385  \hspace{1cm} \textbf{Sample problems}  \hspace{1cm} \text{Name (printed):}

\textit{Show your work. Your answers must be justified to get full credit.}

1. Solve the heat equation $u_t = u_{xx} + u_{yy}$ inside a semicircle of radius 1, and if the initial condition is $u(r, \theta, 0) = \alpha(r, \theta)$ and the boundary conditions are $u(r, 0, t) = 0$, $u(r, \pi, t) = 0$, $u(1, \theta, t) = 0$.

You need to work with the polar coordinate system, for which $u_{xx} + u_{yy} = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$. 

2. Solve the heat equation $u_t = u_{xx} + u_{yy}$ in a two-dimensional rectangular region $0 < x < L$, $0 < y < H$
subject to the initial condition $u(x, y, 0) = \alpha(x, y)$ and the boundary conditions

$$u(0, y, t) = 0, \quad u(L, y, t) = 0; \quad u(x, 0, t) = 0, \quad u(x, H, t) = 0.$$
3. Solve

\[ u_t = u_{xx} + e^{-t} \sin 3x \]

subject to \( u(0, t) = 0, \ u(\pi, t) = 1, \) and \( u(x, 0) = f(x) \).
4. Let \( f(x) \) be defined piecewise on the interval \([0, 1]\) according to

\[
f(x) = \begin{cases} 
  x^2, & 0 < x < \frac{1}{2} \\
  x, & \frac{1}{2} < x < 1 
\end{cases}
\]

(a) Draw the graph of the Fourier Cosine series on \([-3, 3]\). Mark the value for each discontinuous point.

(b) Draw the graph of the Fourier Sine series on \([-3, 3]\). Mark the value for each discontinuous point.

5. Find the solution of the problem

\[
 u_t + 5u_x = 0, \quad x > 0, \quad t > 0.
\]

\[
 u(x, 0) = \sin 2x, \quad x > 0; \quad u(0, t) = e^{-t} - 1, \quad t > 0.
\]