ALTERNATING EVOLUTION SCHEMES FOR HAMILTON–JACOBI EQUATIONS

HAILIANG LIU†, MICHAEL POLLACK†, AND HASEENA SARAN†

Abstract. In this work, we propose a high-resolution alternating evolution (AE) scheme to solve Hamilton–Jacobi equations. The construction of the AE scheme is based on an alternating evolution system of the Hamilton–Jacobi equation, following the idea previously developed for hyperbolic conservation laws. A semidiscrete scheme derives directly from a sampling of this system on alternating grids. Higher order accuracy is achieved by a combination of high order nonoscillatory polynomial reconstruction from the obtained grid values and a time discretization with matching accuracy. Local AE schemes are made possible by choosing the scale parameter $\epsilon$ to reflect the local distribution of waves. The AE schemes have the advantage of easy formulation and implementation and efficient computation of the solution. For the first local AE scheme and the second order local AE scheme with a limiter, we prove the numerical stability in the sense of satisfying the maximum principle. Numerical experiments for a set of Hamilton–Jacobi equations are presented to demonstrate both accuracy and capacity of these AE schemes.

Key words. alternating evolution, Hamilton–Jacobi equations, viscosity solution

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1. Introduction. In this paper, we develop a new alternating evolution (AE) method to solve time-dependent Hamilton–Jacobi (HJ) equations. We describe the designing principle of our new AE scheme through the following form:

$$\phi_t + H(x, \nabla_x \phi) = 0, \quad \phi(x, 0) = \phi_0(x), \quad x \in \mathbb{R}^d, \quad t > 0.$$ 

Here, $d$ is the space dimension, the unknown $\phi$ is scalar, and $H : \mathbb{R}^d \to \mathbb{R}^1$ is a nonlinear Hamiltonian. The HJ equation arises in many applications, ranging from geometrical optics to differential games. These nonlinear equations typically develop discontinuous derivatives even with smooth initial conditions, the solutions of which are nonunique. In this paper, we are only interested in the viscosity solution [7, 6, 29], which is the unique physically relevant solution in some important applications.

The difficulty encountered for the satisfactory approximation of the exact solutions of these equations lies in the presence of discontinuities in the solution derivatives. An important class of finite difference methods for computing the viscosity solution is the class of monotone schemes introduced by Crandall and Lions [8]. Unfortunately, monotone schemes are at most first order accurate. The need for devising more accurate numerical methods for HJ equations has prompted the abundant research in this area in the last two decades, including essentially nonoscillatory (ENO) or weighted ENO (WENO) finite difference schemes (see, e.g., [24, 25, 13, 17, 28, 31]) and central or central-upwind finite difference schemes (see, e.g., [19, 16, 15, 4, 2, 3]), as well as discontinuous Galerkin methods [12, 5, 30, 22].
The preliminary goal of this paper is to design high order finite difference schemes for HJ equations by following the alternating evolution framework introduced in [21] for hyperbolic conservation laws; see local AE schemes developed by Saran and Liu [26]. The global AE scheme [21] shares similar features to central schemes by Y. Liu [23] using overlapping cells. In such a framework, the general setting is to refine the original PDE by an AE system which involves two representatives: \( \{ u, v \} \). In the evolution of \( u \), terms involving spatial derivatives are replaced by \( v \)’s derivatives and augmented by an additional relaxation term \( (v - u)/\epsilon \), which serves to communicate the two representatives. The AE system for scalar hyperbolic conservation laws was shown in [21] to be capable of capturing the exact solution when initially both representatives are chosen as the given initial data. Such a feature allows for a sampling of two representatives over alternating grids. Using this alternating system as a "building base," we apply standard approximation techniques to the AE system: high order accuracy is achieved by a combination of high order nonoscillatory polynomial reconstruction in space and an ODE solver in time with matching accuracy, following those implemented for hyperbolic conservation laws in [26]. Preliminary results on AE schemes for HJ equations were reported in the third author’s thesis [1].

For Hamilton–Jacobi equations, we consider the following alternating evolution system:

\[
\begin{align*}
  u_t + H(x, \nabla_x v) &= \frac{1}{\epsilon} (v - u), \\
  v_t + H(x, \nabla_x u) &= \frac{1}{\epsilon} (u - v).
\end{align*}
\]

Here, \( \epsilon > 0 \) is a scale parameter of the user’s choice. At \( t = 0 \), we take the initial data

\[
  u_0(x) = v_0(x) = \phi_0(x).
\]

We sample the above system over alternating grids when performing the spatial discretization, leading to a class of high-resolution semidiscrete AE schemes. One of the main differences between the formulation of AE schemes for HJ equations and that for conservation laws is that here we use grid point values instead of cell averages.

The semidiscrete schemes thus obtained have the form

\[
\frac{d}{dt} \Phi^n(x) + H(x, \nabla_x p^n_\alpha(\Phi^n(x))) + \frac{1}{\epsilon} \Phi^n_\alpha = \frac{1}{\epsilon} p^n_\alpha(\Phi^n(x)),
\]

where \( p^n_\alpha(\Phi^n(x)) \) is the polynomial reconstruction based on grid values of \( \Phi \) in the domain centered at \( x_\alpha \). The scale parameter \( \epsilon \) and the time step size \( \Delta t \) are chosen to stabilize the time discretization. In the one-dimensional case, the fully discrete AE scheme of first order becomes

\[
\Phi^{n+1}_k = \left( 1 - \frac{\Delta t}{\epsilon} \right) \Phi^n_k + \frac{\Delta t}{\epsilon} \left[ \frac{1}{2} (\Phi^{n+1}_{k+1} + \Phi^{n-1}_{k-1}) - \epsilon H \left( x_k, \frac{\Phi^{n+1}_{k+1} - \Phi^{n-1}_{k-1}}{2\Delta x} \right) \right], \quad \epsilon \geq \Delta t,
\]

which when \( \epsilon = \Delta t \) is taken reduces to the celebrated Lax–Friedrichs scheme

\[
\Phi^{n+1}_k = \frac{1}{2} (\Phi^{n+1}_{k+1} + \Phi^{n-1}_{k-1}) - \epsilon H \left( x_k, \frac{\Phi^{n+1}_{k+1} - \Phi^{n-1}_{k-1}}{2\Delta x} \right).
\]

Thus the class of AE schemes can be viewed as a higher order extension of the Lax–Friedrichs scheme.

To put our study in the proper perspective, we recall that the success of existing finite difference schemes for HJ equations is due to two factors: the local enforcement
HAILIANG LIU, MICHAEL POLLACK, AND HASEENA SARAN

of the equation in question and the nonoscillatory piecewise polynomial interpolation from evolved grid values. The present AE scheme differs from the existing ones in the local enforcement. The ENO/WENO type schemes [24, 25, 13, 17, 28, 31] are based mainly on some local refinement of HJ equations by

\[ u_t + \tilde{H}(x, \phi_x^+, \phi_x^-) = 0, \]

where \( \tilde{H} \) is the numerical Hamiltonian which needs to be carefully chosen to ensure the viscosity solution is captured when \( \phi_x \) becomes discontinuous. Instead, the central type schemes [20, 19, 16, 15, 4, 2, 3] choose to evolve the constructed polynomials in smooth regions such that the Taylor expansion may be used in the scheme derivation. Effort has been made to extend these ideas toward some discontinuous Galerkin methods such as in [12, 5, 30, 22]; however, these extensions either are restricted or involve some further local refinement.

The scheme we design here is different in that it is based on alternatively sampling the AE system with spatial accuracy enhanced by interlaced local interpolations. The procedure discussed in this paper opens a new door to deriving robust finite difference schemes for HJ equations. Also the semidiscrete scheme thus obtained offers an economic approach for achieving a matching accuracy in time. Another attractive feature of the AE scheme is the amount of the leverage in the choice of \( \epsilon \). Indeed, different choices of the scale parameter in such a procedure yield different AE schemes such as local and global AE schemes. For the polynomial construction, we use standard polynomial interpolation with ENO choice of stencils or appropriate limiters to ensure the computed solution is the viscosity solution. Extension of the AE idea to a discontinuous Galerkin setting is under investigation.

It should be noted that even though our AE schemes are derived based on sampling the AE system, we do not solve the system directly. The AE system simply provides a systematic way for developing numerical schemes of both semidiscrete and fully discrete form for the original problem, instead of as an approximation system at the continuous level.

The article is organized as follows. In section 2, we formulate the AE method for one-dimensional HJ equations. We then give a rigorous proof of \( L^\infty \) stability by the AE method of both first and second order in section 3. Extensions to the multidimensional case are given in section 4, including stability for the AE method of first and second order. In section 5, we show numerical results, which include both one- and two-dimensional problems. The results illustrate accuracy, efficiency, and high resolution near kinks. Section 6 ends this paper with our concluding remarks.

2. Alternating evolution methods. Our numerical schemes for HJ equations consist of a semidiscrete formulation based on sampling of the AE system on alternating grids and a fully discrete version by further using an appropriate Runge–Kutta solver.

To illustrate, we start with the one-dimensional HJ equation of the form

\[ \phi_t + H(x, \phi_x) = 0. \]

The “building base” is the following AE system:

\[
\begin{align*}
(2.1) \quad u_t + H(x, u_x) &= \frac{1}{\epsilon}(v - u), \\
(2.2) \quad v_t + H(x, u_x) &= \frac{1}{\epsilon}(u - v).
\end{align*}
\]
2.1. Semidiscrete formulation. Consider a uniform discretization \( \{ x_k, k \in \mathbb{Z} \} \) with grid size \( \Delta x \). We sample the AE approximate system (2.1)–(2.2) at the even and odd grid points, respectively, to obtain

\[
\begin{align*}
\frac{d}{dt} u_{2i}(t) &= -\frac{1}{\epsilon} u_{2i} + \frac{1}{\epsilon} L[v](x_{2i}, t), \\
\frac{d}{dt} v_{2i+1}(t) &= -\frac{1}{\epsilon} v_{2i+1} + \frac{1}{\epsilon} L[u](x_{2i+1}, t),
\end{align*}
\]

where

\[
L[\Phi](x, t) = \Phi - \epsilon H(x, \Phi).
\]

We assume that we have computed the solution at \( t \), denoted by

\[
\Phi_k = \begin{cases} u_k, & k = 2i, \\
v_k, & k = 2i + 1.
\end{cases}
\]

To update each grid value \( \Phi_k \) with a high order of accuracy, we construct a nonoscillatory polynomial approximation \( p_k[\Phi](x) \) using grid point values \( \{x_{k+li}\} \) for some \( i = 1, 2, \ldots \). The numerical approximation for \( L \) is then realized by \( L[p_k[\Phi]](x, t) \), which leads to the following semidiscrete scheme:

\[
\frac{d}{dt} \Phi_k(t) = -\frac{1}{\epsilon} \Phi_k + \frac{1}{\epsilon} L_k[\Phi](t),
\]

where

\[
L_k[\Phi](t) = p_k[\Phi](x_k) - \epsilon H(x_k, \partial_x p_k[\Phi](x_k)).
\]

The fully discrete scheme follows from applying an appropriate Runge–Kutta solver to (2.3). For a computational domain \([a, b]\) with \( x_0 = a, x_N = b \), and \( \Delta x = (b - a)/N \), we summarize the algorithm as follows.

**Algorithm 2.1.**

1. Initialization: at any node \( x_k \), compute the initial data as \( \Phi_k^0 = \phi_0(x_k) \), \( k = 0, 1, \ldots, N \).
2. Polynomial construction: from \( \{ \Phi_{k+li}\} \) construct \((r + 1)\)th order nonoscillatory polynomial \( p_k[\Phi](x) \) and \( r \)th order polynomial \( \partial_x p_k[\Phi](x) \) on \( I_k, k = 1, 2, \ldots, N \), then sample at \( x_k \) to get

\[
L_k[\Phi] = p_k[\Phi](x_k) - \epsilon H(x_k, \partial_x p_k[\Phi](x_k)), \quad k = 1, 2, \ldots, N.
\]
3. Evolution: obtain \( \Phi^{n+1} \) from \( \Phi^n \) by the following TVD Runge–Kutta type procedure [25]:

\[
\Phi_k^{(l)} = \sum_{i=0}^{l-1} \alpha_{li} \Phi_k^{(i)} - \frac{\beta_l}{\epsilon} \left[ \Phi_k^{(i)} + L_k[\Phi^{(i)}] \right], \quad l = 1, \ldots, r,
\]

\[
\Phi_k^{(0)} = \Phi_k^{(n)}, \quad \Phi_k^{n+1} = \Phi_k^{(r)}.
\]

In the AE schemes up to the third order, \( \epsilon \) is chosen such that the stability condition,

\[
\Delta t \leq \epsilon \leq Q \frac{\Delta x}{\max |H_p(x, p)|},
\]

is satisfied. The choice of \( Q \) depends on the order of the scheme; see (3.1) and (3.5) in section 3.
2.2. Fully discrete AE schemes. We now present AE schemes of the first, second, and third order for HJ equations and then discuss advantages of using a local parameter $\epsilon = c_j^n$.

2.2.1. First order scheme. We start with a linear interpolant at nodes $\{x_{k-1}, x_{k+1}\}$,

$$p^1_k[\Phi](x) = \Phi_{k-1} + s_k(x - x_{k-1}), \quad s_k = \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x},$$

and sample at $x_k$,

$$p^1_k[\Phi](x_k) = \frac{\Phi_{k+1} + \Phi_{k-1}}{2} \quad \text{and} \quad \partial_x p^1_k[\Phi](x_k) = s_k = \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x}.$$

This, when combined with a forward Euler in time discretization, gives the first order numerical scheme

$$\Phi^n_{k+1} = (1 - \kappa)\Phi^n_k + \kappa \left[ \frac{1}{2}(\Phi^n_{k+1} + \Phi^n_{k-1}) - \epsilon H \left( x_k, \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x} \right) \right],$$

where $\kappa := \frac{\Delta t}{\max\{|H_j(x)|\}} < 1$. When $\epsilon \leq \frac{\Delta t}{\max\{|H_j(x)|\}}$, this forms a class of monotone schemes, with the celebrated Lax–Friedrichs scheme being a special case of $\kappa = 1$.

2.2.2. Second order scheme. A second order continuous, piecewise quadratic reconstruction gives

$$p^2_k[\Phi](x) = \Phi_{k-1} + s_k(x - x_{k-1}) + \frac{s'_k}{2}(x - x_{k-1})(x - x_{k+1}),$$

where $s_k$ is the first numerical derivative defined as before and $s'_k$ is an approximation to the exact second derivative $\partial_{xx}$. We then obtain

$$p^2_k[\Phi](x_k) = \frac{\Phi_{k+1} + \Phi_{k-1}}{2} - \frac{s'_k}{2}(\Delta x)^2 \quad \text{and} \quad \partial_x p^2_k[\Phi](x_k) = s_k = \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x},$$

which when combined with the second order Runge–Kutta method (Heun’s method) gives

$$\Phi^n_{k+1} = (1 - \kappa)\Phi^n_k + \kappa L_k[\Phi^n],$$

where

$$L_k[\Phi] = \left[ \frac{1}{2}(\Phi^n_{k-1} + \Phi^n_{k+1}) - \frac{(\Delta x)^2}{2} s_k \right] - \epsilon H \left( x_k, \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x} \right).$$

The nonoscillatory nature of the scheme is realized by the choice of $s'_k$. We follow the ENO interpolation technique [11, 25] to select the “smoother” polynomial candidates. For a second order approximation, we choose from two quadratic interpolants which use $\{x_{k-3}, x_{k-1}, x_{k+1}\}$ and $\{x_{k-1}, x_{k+1}, x_{k+3}\}$, respectively, following

$$s'_k = M \left\{ \frac{s_{k+2} - s_k}{2\Delta x}, \frac{s_k - s_{k-2}}{2\Delta x} \right\},$$

where $M(a, b) = a$ if $|a| \leq |b|$ and $b$ if $|b| < |a|$.
2.2.3. Third order scheme. We formulate the third order scheme by using a cubic polynomial interpolant with the ENO selection. For \( x^* = x_{k-3} \) or \( x^* = x_{k+3} \) determined through the ENO selection (2.8), a cubic polynomial interpolant is given as

\[
p^3_k[\Phi](x) = \Phi_{k-1} + s_k(x - x_{k-1}) + \frac{s_k'}{2}(x - x_{k-1})(x - x_{k+1}) + \frac{s_k''}{6}(x - x_{k-1})(x - x_{k+1})(x - x^*),
\]

where the ENO selection requires that

\[
s''_k = M\left\{\frac{s_{k+2} - s_k'}{2\Delta x}, \frac{s_k' - s_{k-2}'}{2\Delta x}\right\}.
\]

The obtained polynomial and its derivative when evaluated at \( x_k \) give

\[
p^3_k[\Phi](x_k) = \frac{1}{2}(\Phi_{k+1} + \Phi_{k-1}) - \frac{s_k'}{2}(\Delta x)^2 - \frac{s_k''}{6}(\Delta x)^2(x_k - x^*),
\]

\[
\partial_x p^3_k[\Phi](x_k) = \frac{\Phi_{k+1} - \Phi_{k-1}}{2\Delta x} - \frac{s_k''}{6}(\Delta x)^2,
\]

which when combined with the third order Runge–Kutta method gives

\[
\Phi_k^{(1)} = (1 - \kappa)\Phi_k^n + \kappa L_k[\Phi^n],
\]

\[
\Phi_k^{(2)} = \frac{3}{4}\Phi_k^n + \frac{1}{4}(1 - \kappa)\Phi_k^{(1)} + \frac{1}{4}\kappa L_k[\Phi^{(1)}],
\]

\[
\Phi_k^{n+1} = \frac{1}{3}\Phi_k^n + \frac{2}{3}(1 - \kappa)\Phi_k^{(2)} + \frac{2}{3}\kappa L_k[\Phi^{(2)}]
\]

with

\[
L_k[\Phi] = \left[\frac{1}{2}(\Phi_{k-1} + \Phi_{k+1}) - \frac{s_k'}{2}(\Delta x)^2 - \frac{s_k''}{6}(\Delta x)^2(x_k - x^*)\right] - \epsilon H\left(x_k, s_k - \frac{s_k''}{6}(\Delta x)^2\right).
\]

Remark 2.2. The above ENO selection procedure can be applied hierarchically to any higher order polynomial approximation. Also, to make stencils more compact, mixed grids \( \{\Phi_{k+1}\} \) instead of even or odd grids \( \{\Phi_{k+2i-1}\} \) may be used in the polynomial construction.

2.3. Local AE schemes. In practice, it is preferred to use local AE schemes, in which local speeds instead of global speeds are explored in the scheme formulation for \( \epsilon \). The notation \( M_k \) is used to denote the range

\[
M_k = \left[\min_{I_k} \partial_x p_k[\Phi](x), \max_{I_k} \partial_x p_k[\Phi](x)\right],
\]

where \( I_k = [x_{k-1}, x_{k+1}] \). Depending on the way \( \epsilon_k \) is defined, we can formulate two local AE schemes:
1. Local AE1 scheme. We choose $\epsilon_k$ such that

$$\epsilon_k \leq Q \frac{\Delta x}{\max_{p \in M_k, x \in I_k} |H_p(x, p)|}.$$  

(2.10)

2. Local AE2 scheme. We choose $\epsilon_k$ such that

$$\epsilon_k \leq Q \frac{\Delta x}{\max_{x \in I_k} \left| \frac{1}{|M_k|} \int_{M_k} |H_p(x, p)| \, dp \right|}.$$  

(2.11)

In both cases we want the time step $\Delta t < \min_k \epsilon_k$, since the stability conditions require that $\kappa = \frac{\Delta t}{\epsilon_k} < 1$. $Q$ is a factor that is dependent on the order of the scheme and the stability conditions to be presented in section 3 provide the range of values $Q$ can take.

3. Stability analysis. In this section we show that the first and second order schemes are nonoscillatory in the sense of satisfying the maximum principle. Let $\Phi^n$ be a computed solution and $|\Phi^n|_\infty = \max_k \{|\Phi^n_k|\}$ define the standard $l^\infty$ norm. For simplicity we present the stability results only for the case $H(x, p) = H(p)$.

We first prove stability for global AE schemes. The proofs for local schemes are similar; only $\epsilon$ needs to be considered as defined locally instead of globally.

**Theorem 3.1.** Let $\Phi$ be computed from the first order AE scheme (2.4) for the HJ equation $\phi_t + H(\phi_x) = 0$. If

$$\epsilon \frac{\Delta x}{\max |H'(\cdot)|} \leq 1 \quad \text{and} \quad \Delta t < \epsilon,$$  

(3.1)

then

$$\min_k \Phi^n_k \leq \Phi^{n+1}_k - H(0) \Delta t \leq \max_k \Phi^n_k, \quad n \in \mathbb{N}.$$  

(3.2)

**Proof.** The scheme (2.4) can be written as

$$\Phi^{n+1}_k = (1 - \kappa) \Phi^n_k + \kappa L_k[\Phi], \quad L_k[\Phi] = \frac{\Phi^n_{k-1} + \Phi^n_{k+1}}{2} - \epsilon H \left( \frac{\Phi^n_{k+1} - \Phi^n_{k-1}}{2\Delta x} \right).$$

When $\frac{\Delta x}{\max |H'|} \leq 1$, $L_k[\Phi]$ becomes nondecreasing in both $\Phi_{k-1}$ and $\Phi_{k+1}$, we have

$$\min \Phi_k - \epsilon H(0) = L_k[\min \Phi_k] \leq L_k[\Phi] \leq L_k[\max \Phi_k] = \max \Phi_k - \epsilon H(0).$$

Note that for $\kappa < 1$, $\Phi^{n+1}_k$ is a convex combination of $\Phi^n_k$ and $L_k[\Phi^n]$, therefore the claimed estimate (3.2) follows.

Without loss of generality we can assume $H(0) = 0$ (this can be made so by a transform of $\phi - H(0)t$ in the original equation), so that the estimate (3.2) becomes

$$|\Phi^{n+1}|_{\infty} \leq |\Phi^n|_{\infty}.$$
Fig. 1. Comparison of three different approximations for $s'_k$, ENO from (2.8) and MM from (3.3) and (3.4), when second order scheme is applied to Example 5.3 with $T = 1$, $N = 80$, $\Delta t = 0.9\epsilon$.

Such an estimate for second order AE schemes still holds if some stricter limiter on numerical derivatives $s'_k$ is taken. As is known, an alternative limiter for the second order scheme is the minmod limiter

$$(3.3)\quad s'_k = \text{MM} \left\{ \frac{s_{k+1} - s_k}{\Delta x}, \frac{s_k - s_{k-1}}{\Delta x} \right\}.$$ 

Here $\text{MM}$ denotes the minmod nonlinear limiter

$$\text{MM}\{x_1, x_2, \ldots\} = \begin{cases} 
\min_j \{x_j\} & \text{if } x_j > 0 \ \forall j, \\
\max_j \{x_j\} & \text{if } x_j < 0 \ \forall j, \\
0 & \text{otherwise.}
\end{cases}$$

We modify this limiter to obtain

$$(3.4)\quad s'_k = \frac{1}{\Delta x} (\text{MM}\{s_{k+1}, s_k\} - \text{MM}\{s_k, s_{k-1}\}).$$

A comparison of the ENO selection and the minmod limiters from (3.3) and (3.4) is shown in Figure 1 for Example 5.3. As one can see, the ENO approximation provides the most accurate results for the reference solution.

In order to establish the stability result, $s'_k$ will be chosen as the stricter limiter from (3.4).

**Theorem 3.2.** Let $\Phi^n$ be computed from the second order AE scheme (2.5) for HJ equations with $H(0) = 0$ and with slopes $s'_k$ defined as in (3.4). If

$$(3.5)\quad \frac{\epsilon}{\Delta x} \max |H'(\cdot)| \leq \frac{1}{2} \quad \text{and} \quad \Delta t < \epsilon$$

hold, then

$$(3.6)\quad |\Phi^{n+1}|_\infty \leq |\Phi^n|_\infty, \quad n \in \mathbb{N}.$$ 

**Proof.** It suffices to show that

$$(3.7)\quad |L[\Phi^n]|_\infty \leq |\Phi^n|_\infty,$$
so that when $\kappa < 1$,

$$|\Phi^*|_\infty \leq (1 - \kappa)|\Phi^0|_\infty + \kappa|\Phi^n|_\infty \leq |\Phi^n|_\infty.$$  

Then, from (2.6), it follows that

$$|\Phi^{n+1}|_\infty \leq \frac{1}{2}|\Phi^n|_\infty + \left(\frac{1 - \kappa}{2}\right)|\Phi^*|_\infty + \frac{\kappa}{2}|L[\Phi^*]|_\infty$$

from which the maximum principle as claimed in (3.6) follows.

We now prove (3.7). From (2.7) we have that

$$L_k[\Phi^n] = \frac{1}{2}(\Phi^n_{k+1} + \Phi^n_{k-1}) - \frac{s'_k}{2}(\Delta x)^2 - \epsilon H\left(\frac{\Phi^n_{k+1} - \Phi^n_{k-1}}{2\Delta x}\right).$$

Notice that $H(0) = 0$ and define

$$\beta^*_k := \begin{cases} 
-\frac{(\Delta x)^2}{\Phi^n_{k+1} - \Phi^n_{k-1}} & \text{if } \Phi^n_{k+1} \neq \Phi^n_{k-1}, \\
0 & \text{if } \Phi^n_{k+1} = \Phi^n_{k-1}.
\end{cases}$$

Hence,

$$L_k[\Phi^n] = \frac{1}{2}(\Phi^n_{k+1} + \Phi^n_{k-1}) - \frac{\beta^n_k}{2}(\Phi^n_{k+1} - \Phi^n_{k-1}) - \frac{\epsilon}{2\Delta x}H'(\cdot)(\Phi^n_{k+1} - \Phi^n_{k-1})$$

$$= \frac{1}{2}\left(1 + \frac{\beta^n_k}{\Delta x}H'(\cdot)\right)\Phi^n_{k+1} + \frac{1}{2}\left(1 - \frac{\beta^n_k}{\Delta x}H'(\cdot)\right)\Phi^n_{k-1},$$

where $(\cdot)$ is an unspecified intermediate value between 0 and $s_k$. The minmod property on the second numerical derivative $s'_k$ leads to

$$s'_k \leq \frac{|s_k|}{\Delta x} = \frac{1}{2(\Delta x)^2}|\Phi^n_{k+1} - \Phi^n_{k-1}|,$$

and hence $\beta^n_k \leq \frac{1}{\Delta x}$. This together with the condition $\frac{\epsilon}{\Delta x}\max|H'| \leq \frac{1}{2}$ ensures that all the coefficients on the right of (3.8) are nonnegative and we can take the maximum norm to obtain

$$|L[\Phi^n]|_\infty \leq |\Phi^n|_\infty,$$

as claimed. 

4. The multidimensional case. By similar procedures we can construct AE schemes for multidimensional HJ equations:

$$\phi_t + H(x, \nabla_x \phi) = 0, \quad x \in \mathbb{R}^d.$$  

We start with the AE formulation

$$u_t + \frac{1}{\epsilon}u = \frac{1}{\epsilon}v - H(x, \nabla_x v).$$

(4.1)

Let $\{x_\alpha\}$ be uniformly distributed grids in $\mathbb{R}^d$. Consider $I_\alpha$ to be a hypercube centered at $x_\alpha$ with vertices at $x_{\alpha \pm 1}$ where the number of vertices is $2^d$. 
We consider two different constructions of AE schemes for multidimensions. For the first type, given grid values \( \{ \Phi_\alpha \} \), we construct a continuous, piecewise polynomial \( p_\alpha[\Phi](x) \in P_r \) defined in \( I_\alpha \) such that
\[
p_\alpha[\Phi](x_{\alpha \pm 1}) = \Phi_{\alpha \pm 1}.
\]
These polynomials are then put into the right-hand side of (4.1). Here \( P_r \) denotes a linear space of all polynomials of degree at most \( r \) in all \( x_i \):
\[
P_r := \left\{ p \mid p(x) = \sum_{0 \leq \beta \leq r} a_\beta x^\beta, \ 1 \leq i \leq d, \ a_\beta \in \mathbb{R} \right\}.
\]
Note \( \dim(P_r) = (r + 1)^d \).

Sampling the AE system (4.1) at \( x_\alpha \), which is the common vertex of \( I_{\alpha \pm 1} \), we obtain the semidiscrete AE scheme
\[
dt \Phi_{\alpha} + \frac{1}{\epsilon} \Phi_{\alpha} = \frac{1}{\epsilon} L_\alpha[\Phi],
\]
\[
L_\alpha[\Phi] = p_\alpha[\Phi](x_\alpha) - \epsilon H(x_k, \nabla x p_\alpha[\Phi](x_\alpha)).
\]
The second type will involve a dimension-by-dimension approach. For the two-dimensional case, we construct a polynomial \( p_{j,k} \) in the \( x \)-direction and \( q_{j,k} \) in the \( y \)-direction in a similar fashion to the one-dimensional case. The AE formulation for two dimensions then becomes
\[
u_t + \frac{1}{\epsilon} \nu = \frac{1}{\epsilon} v - H(\partial_x p_{j,k} [\nu], \partial_y q_{j,k} [\nu]).
\]
An average of the \( p \) and \( q \) polynomials will be used on the \( v \) term so that the semidiscrete AE scheme becomes
\[
dt \Phi_{j,k} + \frac{1}{\epsilon} \Phi_{j,k} = \frac{1}{\epsilon} L_{j,k}[\Phi],
\]
where
\[
L_{j,k} = \frac{p_{j,k}[\Phi](x_j, y_k) + q_{j,k}[\Phi](x_j, y_k)}{2} - \epsilon H(\partial_x p_{j,k} [\Phi](x_j, y_k), \partial_y q_{j,k} [\Phi](x_j, y_k)).
\]
The strong stability-preserving Runge–Kutta method [10] can be used to achieve a time discretization with matching accuracy.

4.1. Averaging construction. We will consider the first type of construction for an AE scheme in multidimensions, which will be referred to as the averaging construction. For simplicity, we now present our AE schemes in the two-dimensional case \( d = 2 \) and use \( x, y \) instead of \( x_1, x_2 \). For mesh sizes \( \Delta x, \Delta y, \Delta t, \Phi_{j,k}^n \) will denote the numerical approximation to the viscosity solution \( \phi(x_j, y_k, t_n) = \phi(j \Delta x, k \Delta y, n \Delta t) \) of
\[
\phi_t + H(\phi_x, \phi_y) = 0.
\]
We shall use the notation
\[
D_x^h \Phi_{j,k} = \frac{\Phi_{j+1,k} - \Phi_{j-1,k}}{2 \Delta x}, \quad D_y^h \Phi_{j,k} = \frac{\Phi_{j,k+1} - \Phi_{j,k-1}}{2 \Delta y}.
\]
for central differences and
\[ A_x \Phi_{j,k} = \frac{\Phi_{j+1,k} + \Phi_{j-1,k}}{2}, \quad A_y \Phi_{j,k} = \frac{\Phi_{j,k+1} + \Phi_{j,k-1}}{2} \]
for averages.

We begin with a continuous linear interpolation over each rectangular \( I_{j,k} \) centered at \((x_j, y_k) = (j \Delta x, k \Delta y)\) with vertices at \((x_{j \pm 1}, y_{k \pm 1})\). In this case the polynomial is uniquely determined by grid values at four vertices of \( I_{j,k} \) satisfying
\[ p_{j,k}[\Phi](x_j \pm 1, y_k \pm 1) = \Phi_{j \pm 1, k \pm 1}. \]

Such an interpolant is given by
\[ p^1_{j,k}[\Phi](x, y) = A_x A_y \Phi_{j,k} + D_x^0 A_y \Phi_{j,k}(x-x_j) + D_y^0 A_x \Phi_{j,k}(y-y_k) + D_x^0 D_y^0 \Phi_{j,k}(x-x_j)(y-y_k). \]
Substitution of this into the right-hand side of (4.2) and use of forward Euler time-discretization yields the first order AE scheme,
\[ \Phi^{n+1}_{j,k} = (1-\kappa)\Phi^n_{j,k} + \kappa L_{j,k}[\Phi^n], \]
\[ L_{j,k}[\Phi] = A_x A_y \Phi_{j,k} - \epsilon H(D_x^0 A_y \Phi_{j,k}, D_y^0 A_x \Phi_{j,k}). \]
The scale parameter \( \epsilon \) is chosen to be
\[ \epsilon \cdot \left( \frac{\max |H_1(\cdot)|}{\Delta x} + \frac{\max |H_2(\cdot)|}{\Delta y} \right) \leq 1. \]
Here, \( H_i(p, q) \) is the partial derivative of \( H \) with respect to the \( i \)-th argument (dependence of \( H \) on \( x \) causes no difficulty). The scheme becomes a local AE scheme with \( \epsilon = \epsilon_{j,k}^n \) when the maximum is taken over \( I_{j,k} \) for all \( \phi = p^1_{j,k}[\Phi^n](x, y) \). Indeed, under this CFL type restriction, \( L_{j,k}[\Phi] \) is nondecreasing in terms of its arguments. Therefore, for \( H(0,0) = 0 \), we have a local bound for \( L_{j,k}[\Phi] \):
\[ \min_{|i-j|=1, |l-k|=1} \{ \Phi_{i,l} \} \leq L_{j,k}[\Phi] \leq \max_{|i-j|=1, |l-k|=1} \{ \Phi_{i,l} \}. \]
This, when combined with \( \kappa \leq 1 \) leads to the local maximum principle
\[ \min_{|i-j|=1, |l-k|=1} \{ \Phi_{i,l} \} \leq \Phi^{n+1}_{j,k} \leq \max_{|i-j|=1, |l-k|=1} \{ \Phi_{i,l} \}, \quad n \in \mathbb{N}. \]

4.2. Stability analysis of second order AE schemes. We start again assuming that \( \Phi^n_{j,k} \) has already been computed. We extend the linear reconstruction \( p^1_{j,k}[\Phi] \) to a continuous, piecewise quadratic polynomial in \( I_{j,k} \). Maintaining the interpolation at four vertices of \( I_{j,k} \) such a polynomial is given by
\[ p^2_{j,k}[\Phi](x, y) = p^1_{j,k}[\Phi](x, y) + \frac{s_{j,k}'}{2}(x-x_{j-1})(x-x_{j+1}) + \frac{s_{k}'}{2}(y-y_{k-1})(y-y_{k+1}), \]
where \( s_{j,k}' \) and \( s_{k}' \) are approximations to the corresponding exact derivatives \( \phi_{xx}(x_j, y_k, t_n) \) and \( \phi_{yy}(x_j, y_k, t_n) \), respectively. The polynomial is not unique, leaving room to impose a nonlinear limiter on the numerical derivatives to ensure the nonoscillatory
AE schemes for Hamilton–Jacobi equations

A133

maximum principle, we fix

\[ s_{j,k} = \text{MM} \left\{ \theta A_y (D_{x}^0)² \Phi_{j+1,k} , A_y (D_{x}^0)² \Phi_{j,k} , \theta A_y (D_{x}^0)² \Phi_{j-1,k} \right\}, \]

\[ s_{j,k}^\prime = \text{MM} \left\{ \theta A_x (D_{y}^0)² \Phi_{j,k+1} , A_x (D_{y}^0)² \Phi_{j,k} , \theta A_x (D_{y}^0)² \Phi_{j,k-1} \right\}. \]

Here, \( \theta \in [1, 2] \) is a parameter that controls numerical dissipation where \( \theta = 1 \) is the most dissipative and \( \theta = 2 \) is the least dissipative (see, e.g., [18]). We now substitute \( p_{j,k}^2 \Phi \) into \( L_{j,k} \Phi \) to obtain the second order semidiscrete AE scheme:

\[
\frac{d}{dt} \Phi_{j,k} + \frac{1}{\epsilon} \Phi_{j,k} = \frac{1}{\epsilon} L_{j,k}[\Phi],
\]

\[
L_{j,k}[\Phi] = A_x A_y \Phi_{j,k} - \frac{s_{j,k}}{2} (\Delta x)^2 - \frac{s_{j,k}^\prime}{2} (\Delta y)^2 - \epsilon H(D_x^0 A_y \Phi_{j,k} , D_y^0 A_x \Phi_{j,k}).
\]

An application of the second order Runge–Kutta solver gives the fully discrete second order AE scheme:

\[
\Phi_{j,k}^n = (1 - \kappa) \Phi_{j,k}^n + \kappa L_{j,k}[\Phi^n],
\]

\[
\Phi_{j,k}^{n+1} = \frac{1}{2} \Phi_{j,k}^n + \left( \frac{1 - \kappa}{2} \right) \Phi_{j,k}^* + \frac{\kappa}{2} L_{j,k}[\Phi^*].
\]

As in the one-dimensional case, for the second order scheme to still satisfy the maximum principle, we fix \( \theta = 1 \) and modify the limiter as

\[
s_{j,k} = D_x^0 \left\{ \text{MM} \{ A_y D_x^0 \Phi_{j+1,k} , A_y D_x^0 \Phi_{j,k} , A_y D_x^0 \Phi_{j-1,k} \} \right\},
\]

\[
s_{j,k} = D_y^0 \left\{ \text{MM} \{ A_x D_y^0 \Phi_{j,k+1} , A_x D_y^0 \Phi_{j,k} , A_x D_y^0 \Phi_{j,k-1} \} \right\}.
\]

As illustrated in Figure 1 for the one-dimensional case, this modified limiter could be inferior to (4.6), (4.7), yet it ensures the maximum-satisfying property of the second order scheme.

THEOREM 4.1. Let \( \Phi^n \) be computed from the second order AE scheme (4.10), (4.11), and (4.9) for HJ equations with \( H(0,0) = 0 \) and slopes \( s_{j,k}^\prime, s_{j,k} \) defined in (4.12)–(4.13). If

\[
\epsilon \cdot \left( \frac{\max |H_1(\nabla \phi)|}{\Delta x} + \frac{\max |H_2(\nabla \phi)|}{\Delta y} \right) \leq \frac{1}{2}
\]

hold, then

\[ |\Phi^{n+1}|_\infty \leq |\Phi^n|_\infty, \quad n \in \mathbb{N}. \]

Proof. It suffices to show

\[ |L[\Phi^n]|_\infty \leq |\Phi^n|_\infty, \]

so that when \( \kappa < 1 \), we have that

\[ |\Phi^{n+1}|_\infty \leq \frac{1}{2} |\Phi^n|_\infty + \left( \frac{1}{2} - \frac{\kappa}{2} \right) |\Phi^*|_\infty + \frac{\kappa}{2} |L[\Phi^*]|_\infty \leq |\Phi^n|_\infty. \]
We now prove (4.2) as follows. Set

\[
\beta'_{j,k} := \begin{cases} 
-(\Delta x)^2 \frac{s'_{j,k}}{D_x A_y \Phi_{j,k}} & \text{if } D_x^0 A_y \Phi_{j,k} \neq 0, \\
0 & \text{if } D_x^0 A_y \Phi_{j,k} = 0
\end{cases}
\]

and

\[
\beta''_{j,k} := \begin{cases} 
-(\Delta y)^2 \frac{s'_{j,k}}{D_y A_x \Phi_{j,k}} & \text{if } D_y^0 A_x \Phi_{j,k} \neq 0, \\
0 & \text{if } D_y^0 A_x \Phi_{j,k} = 0.
\end{cases}
\]

Notice that \( H(0,0) = 0 \), and by taking the Taylor expansion of the Hamiltonian we have

\[
L_{j,k}[\Phi] = A_x A_y \Phi_{j,k} + \left( \frac{\beta'_{j,k}}{2} - \epsilon H_1(\cdot) \right) D_x^0 A_y \Phi_{j,k} + \left( \frac{\beta''_{j,k}}{2} - \epsilon H_2(\cdot) \right) D_y^0 A_x \Phi_{j,k},
\]

where \((\cdot)\) denotes some unspecified intermediate values. Regrouping terms we obtain

\[
L_{j,k}[\Phi] = \frac{1}{4} \left( 1 + \frac{a_1}{\Delta x} + \frac{a_2}{\Delta y} \right) \Phi_{j+1,k+1} + \frac{1}{4} \left( 1 + \frac{a_1}{\Delta x} - \frac{a_2}{\Delta y} \right) \Phi_{j+1,k-1} + \frac{1}{4} \left( 1 - \frac{a_1}{\Delta x} + \frac{a_2}{\Delta y} \right) \Phi_{j-1,k+1} + \frac{1}{4} \left( 1 - \frac{a_1}{\Delta x} - \frac{a_2}{\Delta y} \right) \Phi_{j-1,k-1},
\]

(4.15)

where

\[
a_1 = \frac{\beta'_{j,k}}{2} - \epsilon H_1(\cdot), \quad a_2 = \frac{\beta''_{j,k}}{2} - \epsilon H_2(\cdot).
\]

Due to the minmod limiters defined in (4.12), (4.13), we claim that

\[
|s'_{j,k}| \leq \frac{|A_y D_x^0 \Phi_{j,k}|}{2\Delta x}, \quad |s''_{j,k}| \leq \frac{|A_x D_y^0 \Phi_{j,k}|}{2\Delta y}.
\]

(4.16)

Thereby,

\[
|\beta'_{j,k}| \leq \frac{\Delta x}{2}, \quad |\beta''_{j,k}| \leq \frac{\Delta y}{2},
\]

which when combined with the condition (4.14) yields

\[
\frac{|a_1|}{\Delta x} + \frac{|a_2|}{\Delta y} \leq \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1.
\]

Thus all the coefficients on the right of (4.15) are nonnegative, and \( L_{j,k}[\Phi] \) is a convex combination of four values \( \Phi_{j\pm1,k\pm1} \). This implies

\[
|L[\Phi^n]|_\infty \leq |\Phi^n|_\infty.
\]

Finally we show claim (4.16). For the first inequality, we fix \( k \) and set \( b_j := A_y D_x^0 \Phi_{j,k} \) and \( B_j = MM\{b_{j+1}, b_j, b_{j-1}\} \) so that

\[
s'_{j,k} = \frac{1}{2\Delta x} (B_{j+1} - B_{j-1}).
\]
By the definition of $MM$, we have
\[ |s'_{j,k}| \leq \frac{1}{2\Delta x} \min\{|B_{j-1}|, |B_{j+1}|\} \]
\[ \leq \frac{1}{2\Delta x} \min_{i=\pm 2} |b_j| \]
\[ \leq \frac{1}{2\Delta x} |b_j|. \]
The second inequality in (4.16) can be shown in same fashion.

Using this averaging method, construction of schemes higher than second order will become cumbersome. We therefore adopt a dimension-by-dimension approach that can be easily derived for higher order schemes.

4.3. Dimension-by-dimension approach. We now consider the second construction method of the two-dimensional AE scheme which will be referred to as the dimension-by-dimension approach. We consider interpolated polynomials, $p_{j,k}$ and $q_{j,k}$ in the $x$- and $y$-directions, satisfying
\[ p_{j,k} \Phi(x_{j+1}, y_k) = \Phi_{j+1,k}, \]
\[ q_{j,k} \Phi(x_j, y_{k+1}) = \Phi_{j,y+1} \]
so that
\[ L_{j,k} = \frac{p_{j,k} \Phi(x_j, y_k) + q_{j,k} \Phi(x_j, y_k)}{2} - \epsilon H(\partial_x p_{j,k} \Phi(x_j, y_k), \partial_y q_{j,k} \Phi(x_j, y_k)). \]

4.3.1. First order scheme. For the first order scheme, such an interpolant is given by
\[ p^1_{j,k} \Phi(x, y) = \Phi_{j-1,k} + s^x_{j,k}(x - x_{j-1}), \]
\[ q^1_{j,k} \Phi(x, y) = \Phi_{j,k-1} + s^y_{j,k}(y - y_{k-1}), \]
where
\[ s^x_{j,k} = \frac{\Phi_{j+1,k} - \Phi_{j-1,k}}{2\Delta x}, \quad s^y_{j,k} = \frac{\Phi_{i,k+1} - \Phi_{i,k-1}}{2\Delta y}. \]
Evaluating at the polynomials and their partial derivatives at $(x_j, y_k)$ yields
\[ p^1_{j,k} \Phi(x_j, y_k) = \frac{\Phi_{j+1,k} + \Phi_{j-1,k}}{2}, \quad \partial_x p^1_{j,k} \Phi(x_j, y_k) = s^x_{j,k}, \]
\[ q^1_{j,k} \Phi(x_j, y_k) = \frac{\Phi_{j,k+1} + \Phi_{j,k-1}}{2}, \quad \partial_y q^1_{j,k} \Phi(x_j, y_k) = s^y_{j,k}. \]
Substituting this into (4.17) yields
\[ L_{j,k} \Phi = A_x A_y \Phi_{j,k} - \epsilon H(s^x_{j,k}, s^y_{j,k}) \]
so that
\[ \Phi^{n+1}_{j,k} = (1 - \kappa)\Phi^n_{j,k} + \kappa L_{j,k} [\Phi^n]. \]
Note that the only difference in first order schemes for (4.5) and (4.18) is in the computation of $\phi_x$ and $\phi_y$ for $H(\phi_x, \phi_y)$. 

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4.3.2. Second order scheme. For the second order scheme, the continuous, piecewise interpolating polynomial is given by

\[ p_{j,k}^2[\Phi](x,y) = \Phi_{j-1,k} + s_{j,k}^x(x-x_{j-1}) + \frac{(s_{j,k}^x)'}{2}(x-x_{j-1})(x-x_{j+1}), \]

\[ q_{j,k}^2[\Phi](x,y) = \Phi_{j,k-1} + s_{j,k}^y(y-y_{j-1}) + \frac{(s_{j,k}^y)'}{2}(y-y_{j-1})(y-y_{k+1}), \]

where the approximations to the second derivative are approximated using the ENO interpolation technique,

\[ (s_{j,k}^x)' = M \left\{ \frac{s_{j+2,k}^x - s_{j,k}^x}{2\Delta x}, \frac{s_{j,k}^x - s_{j-2,k}^x}{2\Delta x} \right\}, \]

\[ (s_{j,k}^y)' = M \left\{ \frac{s_{j,k+2}^y - s_{j,k}^y}{2\Delta y}, \frac{s_{j,k}^y - s_{j,k-2}^y}{2\Delta y} \right\}. \]

We then obtain

\[ p_{j,k}^2[\Phi](x_j,y_k) = \Phi_{j+1,k} + \Phi_{j-1,k} - \frac{(s_{j,k}^x)'}{2}(\Delta x)^2, \quad \partial_x p_{j,k}^2[\Phi](x_j,y_k) = s_{j,k}^x, \]

\[ q_{j,k}^2[\Phi](x_j,y_k) = \Phi_{j,k+1} + \Phi_{j,k-1} - \frac{(s_{j,k}^y)'}{2}(\Delta y)^2, \quad \partial_y q_{j,k}^2[\Phi](x_j,y_k) = s_{j,k}^y, \]

so that

\[ L_{j,k}[\Phi] = A_x A_y \Phi_{j,k} - \frac{(s_{j,k}^x)'}{2}(\Delta x)^2 - \frac{(s_{j,k}^y)'}{2}(\Delta y)^2 - \epsilon H(s_{j,k}^x, s_{j,k}^y). \]

Combined with a second order Runge–Kutta method this gives

\[ \Phi_{j,k}^{n+1} = \left(1 - \kappa\right) \Phi_{j,k}^n + \kappa L_{j,k}[\Phi^n], \]

\[ \Phi_{j,k}^{n+1} = \frac{1}{2} \Phi_{j,k}^n + \frac{1 - \kappa}{2} \Phi_{j,k}^* + \frac{\kappa}{2} L_{j,k}[\Phi^*]. \]

Again, the only difference in second order schemes for the two types of two-dimensional AE schemes is in the computation of \(\phi_x\) and \(\phi_y\) for \(H(\phi_x, \phi_y)\).

4.3.3. Third order scheme. The third order scheme formulation is based on the cubic polynomials

\[ p_{j,k}^3[\Phi](x,y) = p_{j,k}^2[\Phi](x,y) + \frac{(s_{j,k}^x)''}{6}(x-x_{j-1})(x-x_{j+1})(x-x), \]

\[ q_{j,k}^3[\Phi](x,y) = q_{j,k}^2[\Phi](x,y) + \frac{(s_{j,k}^y)''}{6}(y-y_{k-1})(y-y_{k+1})(y-y), \]

where

\[ (s_{j,k}^x)'' = M \left\{ \frac{(s_{j+2,k}^x)'}{2\Delta x}, \frac{(s_{j,k}^x)'}{2\Delta x} \right\}, \]

\[ (s_{j,k}^y)'' = M \left\{ \frac{(s_{j,k+2}^y)'}{2\Delta y}, \frac{(s_{j,k}^y)'}{2\Delta y} \right\}. \]
Here, $x^*$ is chosen as the grid point value used in the ENO procedure for $(s_{j,k}^x)'$ so that $x^* = x_{j-3}$ or $x^* = x_{j+3}$. The value $y^*$ is chosen in a similar way. This gives

$$P_j^3[\Phi](x_j, y_k) = \frac{\Phi_{j+1,k} + \Phi_{j-1,k}}{2} - \frac{(s_{j,k}^x)'(\Delta x)^2}{2} - \frac{(s_{j,k}^x)''}{6}(\Delta x)^2(x - x^*),$$

$$\partial_x P_j^3 = s_{j,k}^x - \frac{(s_{j,k}^x)''}{6}(\Delta x)^2,$$

$$q_j^3[\Phi](x_j, y_k) = \frac{\Phi_{j,k+1} + \Phi_{j,k-1}}{2} - \frac{(s_{j,k}^y)'(\Delta y)^2}{2} - \frac{(s_{j,k}^y)''}{6}(\Delta y)^2(y - y^*),$$

$$\partial_y q_j^3 = s_{j,k}^y - \frac{(s_{j,k}^y)''}{6}(\Delta y)^2.$$  

When combined with the third order Runge–Kutta method, this gives

$$\Phi_{j,k}^{(1)} = (1 - \kappa)\Phi_{j,k}^n + \kappa L_{j,k}[\Phi^n],$$

$$\Phi_{j,k}^{(2)} = \frac{3}{4}\Phi_{j,k}^n + \frac{1}{4}(1 - \kappa)\Phi_{j,k}^{(1)} + \frac{1}{4}\kappa L_{j,k}[\Phi^{(1)}],$$

$$\Phi_{j,k}^{n+1} = \frac{1}{3}\Phi_{j,k}^n + \frac{2}{3}(1 - \kappa)\Phi_{j,k}^{(2)} + \frac{2}{3}\kappa L_{j,k}[\Phi^{(2)}],$$

where

$$L_{j,k}[\Phi] = A_x A_y \Phi_{j,k} - \frac{(s_{j,k}^x)'(\Delta x)^2}{2} - \frac{(s_{j,k}^y)'(\Delta y)^2}{2} - \frac{(s_{j,k}^x)''}{6}(\Delta x)^2(x - x^*) - \frac{(s_{j,k}^y)''}{6}(\Delta y)^2(y - y^*)$$

$$- \epsilon H \left(s_{j,k}^x - \frac{(s_{j,k}^x)''}{6}(\Delta x)^2, s_{j,k}^y - \frac{(s_{j,k}^y)''}{6}(\Delta y)^2\right).$$

Since this scheme can be easily derived for higher orders using an ENO procedure for derivative approximations, it will be used for all convergence studies in the numerical results.

**5. Numerical tests.** In this section, we use some model problems to numerically test the first, second, and third order global and local AE schemes. If $\phi$ is the exact solution and $\Phi$ is the computed solution, then the numerical $L^1$ and $L^\infty$ errors in the one-dimensional case are calculated as

$$L^1 \text{ error} = \sum_k |\phi_k - \Phi_k| \Delta x, \quad L^\infty \text{ error} = \max_k |\phi_k - \Phi_k|.$$

We call $Q$ in (2.10) and (2.11) the CFL type number for our numerical tests. In calculating the numerical derivatives in the approximation, ENO constructions are used in all one-dimensional tests. The $\theta$-refined minmod limiters (4.6), (4.7) from the averaging construction scheme are used in two-dimensional tests of the second order scheme while all convergence studies use the dimension-by-dimension approach. A performance comparison of different limiters is illustrated in Figure 1. In what follows we use $N$ to denote the number of intervals the domain is divided into, i.e., $\Delta x = (b - a)/N$ for domain $[a, b]$. The same notation will be used in the two-dimensional case as long as the same partition number is used in both $x$- and $y$-directions.
Table 1

The $L^1$ and $L^\infty$ errors for convex Hamiltonian, Example 5.1, using $N$ equally spaced cells for global and local AE first order schemes at $T = \frac{5\pi}{2}$ when the solution is smooth.

<table>
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<th>$N$</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
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<td>1.866756E-01</td>
<td>1.380235E-01</td>
<td>1.2119</td>
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<td>1.304142E-02</td>
<td>6.012093E-02</td>
<td>0.9921</td>
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<tr>
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<td>AE2</td>
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<td>3.248349E-02</td>
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<td>6.612020E-02</td>
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Example 5.1. We first test the numerical accuracy of the designed schemes by using the Hamilton–Jacobi equation with convex Hamiltonian and with smooth initial data. The equation is

$$\Phi_t + \frac{(\Phi_x + 1)^2}{2} = 0, \quad -1 \leq x \leq 1,$$

with initial data

$$\Phi(x, 0) = -\cos \pi x.$$

For numerical accuracy results, the reference solution is computed using a fifth order WENO [14, 27], fourth order TVD Runge–Kutta [10, 9], and $N = 10240$ with periodic boundary conditions. In Tables 1–3, the numerical accuracy results for the first, second, and third order global and local schemes are presented when the solution is smooth (time $T = \frac{5\pi}{2}$). All the proposed methods give the desired order of accuracy.

The solution becomes discontinuous at time $\frac{1}{\pi}$, and in Figures 2–4 we plot the solutions at time $T = \frac{1}{\pi}$. We can see that local schemes give better numerical resolution than the global schemes since they have smaller numerical viscosity. For the numerical experiments the CFL number is taken to be 0.9 and $\Delta t = 0.9\epsilon$.

Example 5.2. The following example is with a nonconvex Hamiltonian:

$$\Phi_t + \frac{1}{4}(\Phi_x^2 - 1)(\Phi_x^2 - 4) = 0, \quad -1 \leq x \leq 1,$$

$$\Phi(x, 0) = -2|x|.$$
### Table 2
The $L^1$ error for convex Hamiltonian, Example 5.1, using $N$ equally spaced cells for global and local AE second order schemes at $T = \frac{3\pi}{2}$ when the solution is smooth.

<table>
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<th>$L^1$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
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### Table 3
The $L^1$ error for convex Hamiltonian, Example 5.1, using $N$ equally spaced cells for global and local AE third order schemes at $T = \frac{3\pi}{2}$ when the solution is smooth.

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<th>$L^1$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
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This test problem is used to show resolution of discontinuities when the initial data has discontinuous derivatives. As in the previous problem, the reference solution is computed using a fifth order WENO [14, 27], fourth order TVD Runge–Kutta [10, 9], and $N = 10240$. In testing our AE methods, the CFL number used for all order schemes was 0.9. Linear extension boundary conditions are used for the numerical tests. In Figures 5–7, we can clearly see that the local AE2 scheme gives the best numerical accuracy, followed by local AE1 and the global AE for first, second, and third order methods.

Example 5.3. The one-dimensional Eikonal equation is

$$\Phi_t + |\Phi_x| = 0, \quad 0 \leq x \leq 2\pi,$$

with initial data

$$\Phi(x, 0) = \sin x.$$
If $0 \leq t \leq \pi/2$,
\[
\Phi(x,t) = \begin{cases} 
\sin(x-t) & \text{if } 0 \leq x \leq \pi/2, \\
\sin(x+t) & \text{if } \pi/2 \leq x \leq 3\pi/2 - t, \\
-1 & \text{if } 3\pi/2 - t \leq x \leq 3\pi/2 + t, \\
\sin(x-t) & \text{if } 3\pi/2 + t \leq x \leq 2\pi.
\end{cases}
\]

If $\pi/2 \leq t \leq \pi$,
\[
\Phi(x,t) = \begin{cases} 
-1 & \text{if } 0 \leq x \leq t - \pi/2, \\
\sin(x-t) & \text{if } t - \pi/2 \leq x \leq \pi/2, \\
\sin(x+t) & \text{if } \pi/2 \leq x \leq 3\pi/2 - t, \\
-1 & \text{if } 3\pi/2 - t \leq x \leq 2\pi.
\end{cases}
\]

If $t \geq \pi$,
\[
\Phi(x,t) = -1.
\]

Note that local and global schemes are all the same for this example since $|H_p| = 1$.
$T = 1$ in Figure 8. For the viscosity solution, there is a shock wave in $\phi_x$ at $x = \pi/2$ and a rarefaction wave at $x = 3\pi/2$. The CFL number used on the numerical tests is 0.7.

**Example 5.4.** The two-dimensional linear HJ equation with variable coefficients is

$$\phi_t - y\phi_x + x\phi_y = 0 \tag{5.1}$$

with initial condition

$$\phi_0(x, y) = \begin{cases} 
0 & \text{if } 0.3 \leq r, \\
0.3 - r & \text{if } 0.1 \leq r \leq 0.3, \\
0.2 & \text{if } r \leq 0.1,
\end{cases}$$

where $r = \sqrt{(x - 0.4)^2 + (y - 0.4)^2}$. The computational domain is $[-1, 1]^2$ and we impose periodic boundary conditions. The exact solution can be expressed as

$$\phi(x, y, t) = \phi_0(x \cos(t) + y \sin(t), -x \sin(t) + y \cos(t)). \tag{5.2}$$
AE SCHEMES FOR HAMILTON–JACOBI EQUATIONS

Fig. 5. Comparison of plots for HJ equation with Riemann initial data, Example 5.2, at discontinuity on $[-1,1]$, $T = 1$, $N = 80$, $\Delta t = 0.8\epsilon$, first order scheme.

The results for the second order schemes at time $t = 2\pi$ are shown in Figure 9. We choose $\theta = 2$ in the minmod computation for the approximations $s_{j,k}^i$, $s_{j,k}^{\prime}$ and let $\Delta t = .95\epsilon$. Here, we look along the line $x = y$ and compare the global AE scheme to the local AE1 scheme for various mesh sizes. We can see that the local AE1 scheme provides better resolution to the exact solution for the same number of grid points as the global AE scheme.

**Example 5.5.** We solve the two-dimensional Burgers equation using first and second order global schemes by the dimension-by-dimension approach:

$$\phi_t + \left(\phi_x + \phi_y + 1\right)^2 = 0$$

with initial data

$$\phi(x, y, 0) = -\cos(x + y).$$

The computational domain is $[0, 2\pi]^2$ and we impose periodic boundary conditions. The solution is still smooth at time $t = 0.1$ and we test the order of accuracy at this time in Tables 4 and 5. For the first order and second order schemes, we get the expected order in all norms. A convergence test for the third order scheme will be applied to the problem with a nonconvex Hamiltonian (see Example 5.7).
Example 5.6.

\[ \phi_t + \sqrt{\phi_x^2 + \phi_y^2} + 1 = 0 \]

with smooth initial data

\[ \phi(x, y, 0) = \frac{1}{4} (\cos(2\pi x) - 1)(\cos(2\pi y) - 1) - 1 \]

and computational domain \([0, 1]^2\) and periodic boundary conditions. The results at time \(t = 0.6\) are shown in Figure 10. The AE scheme provides high resolution in the formation of the singularity.

Example 5.7. We now consider an example with a nonconvex Hamiltonian:

\[ \phi_t - \cos(\phi_x + \phi_y + 1) = 0 \]

with initial data

\[ \phi(x, y, 0) = -\cos\left(\frac{\pi(x + y)}{2}\right) \]

on the computation domain \([-2, 2]^2\) with periodic boundary conditions. At time \(t = \frac{0.5}{\pi}\), the solution is still smooth and we test the order of accuracy for second
Fig. 7. Comparison of plots for HJ equation with Riemann initial data, Example 5.2, at discontinuity on $[-1,1]$, $T = 1$, $N = 80$, $\Delta t = 0.8\epsilon$, third order scheme.

Fig. 8. Comparison of plots for Eikonal equation, Example 5.3, at discontinuity on $[0,2\pi]$, $T = 1$, $N = 80$, $\Delta t = 0.9\epsilon$, first, second, and third order schemes.
and third order schemes using the dimension-by-dimension approach. The results in Tables 6 and 7 show that we get the desired order in all norms. At time $t = \frac{1}{\pi}$, the solution develops a discontinuous derivative and results at time $t = \frac{1.6}{\pi}$ are shown in Figure 11.

6. Concluding remarks. In this work, we have developed several AE schemes based on the alternating evolution approximation to HJ equations, motivated by our prior work on AE schemes for hyperbolic conservation laws [21, 26]. The AE system captures the exact solution when initially both components are chosen as the given initial data for the HJ equation. We numerically approximate the HJ equation by sampling two components of the AE system over alternating grids. High order accuracy is achieved by a combination of high order nonoscillatory polynomial reconstruction from the obtained grid values and an ODE solver in time discretization.


Table 4

$L^1$, $L^2$, and $L^\infty$ comparison for Example 5.5 at time $t = 0.1$ using first order global AE scheme.

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<tr>
<th>N</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^2$ error</th>
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<td>AE</td>
<td>0.666738528</td>
<td>1.080099099</td>
<td>0.118422976</td>
<td>0.98536682</td>
<td>0.068490748</td>
<td></td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>0.287707844</td>
<td>1.21251654</td>
<td>0.051304179</td>
<td>1.206800783</td>
<td>0.014630959</td>
<td></td>
</tr>
</tbody>
</table>

Table 5

$L^1$, $L^2$, and $L^\infty$ comparison for Example 5.5 at time $t = 0.1$ using second order global AE scheme.

<table>
<thead>
<tr>
<th>N</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^2$ error</th>
<th>$L^2$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>AE</td>
<td>5.487583632</td>
<td>0.913422858</td>
<td>0.214771031</td>
<td>0.214771031</td>
<td></td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>AE</td>
<td>1.584797823</td>
<td>1.791872218</td>
<td>0.27577215</td>
<td>0.727806122</td>
<td>0.070029702</td>
<td>1.61760554</td>
</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>0.39345259</td>
<td>2.010037092</td>
<td>0.071300662</td>
<td>1.951489386</td>
<td>0.019686339</td>
<td>1.830772081</td>
</tr>
<tr>
<td>160</td>
<td>AE</td>
<td>0.095952595</td>
<td>2.035796076</td>
<td>0.01763365</td>
<td>2.01558435</td>
<td>0.004979134</td>
<td>1.983228179</td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>0.022627344</td>
<td>2.084254579</td>
<td>0.004184165</td>
<td>2.075319525</td>
<td>0.001182064</td>
<td>2.074586915</td>
</tr>
</tbody>
</table>

Fig. 10. Surface and contour plots at time $t = 0.6$ for Example 5.6.

Table 6

$L^1$, $L^2$, and $L^\infty$ comparison for Example 5.7 at time $t = 0.5$ using second order dimension-by-dimension global AE scheme.

<table>
<thead>
<tr>
<th>N</th>
<th>Scheme</th>
<th>$L^1$ error</th>
<th>$L^1$ order</th>
<th>$L^2$ error</th>
<th>$L^2$ order</th>
<th>$L^\infty$ error</th>
<th>$L^\infty$ order</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>AE</td>
<td>0.3807545007</td>
<td>0.0961004734</td>
<td>0.0347144173</td>
<td>0.0347144173</td>
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</tr>
<tr>
<td>40</td>
<td>AE</td>
<td>0.092619682</td>
<td>2.0394659222</td>
<td>0.0251604319</td>
<td>1.933886472</td>
<td>0.0102422250</td>
<td>1.7610057921</td>
</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>0.024684717</td>
<td>1.9805993682</td>
<td>0.0064820603</td>
<td>1.956632356</td>
<td>0.0027617606</td>
<td>1.8908642702</td>
</tr>
<tr>
<td>160</td>
<td>AE</td>
<td>0.0058630945</td>
<td>2.0009806787</td>
<td>0.0016286995</td>
<td>1.9927319161</td>
<td>0.0007481906</td>
<td>1.8841152186</td>
</tr>
<tr>
<td>320</td>
<td>AE</td>
<td>0.001448791</td>
<td>2.0247129301</td>
<td>0.0004013811</td>
<td>2.0206757509</td>
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<td>1.9576475279</td>
</tr>
<tr>
<td>640</td>
<td>AE</td>
<td>0.0003416173</td>
<td>2.0764961041</td>
<td>0.0000952364</td>
<td>2.0753878851</td>
<td>0.0000465560</td>
<td>2.0487171898</td>
</tr>
</tbody>
</table>
Table 7

<table>
<thead>
<tr>
<th>(N)</th>
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<th>(L_1 ) error</th>
<th>(L_1 ) order</th>
<th>(L_2 ) error</th>
<th>(L_2 ) order</th>
<th>(L_\infty ) error</th>
<th>(L_\infty ) order</th>
</tr>
</thead>
<tbody>
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<td>0.0822233386</td>
<td>0.0445854569</td>
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</tr>
<tr>
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<td>AE</td>
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<td>2.2442061801</td>
<td>0.0208873181</td>
<td>0.0140420171</td>
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</tr>
<tr>
<td>80</td>
<td>AE</td>
<td>0.0122271144</td>
<td>2.3162931238</td>
<td>0.0044368628</td>
<td>2.2350153966</td>
<td>0.0034367453</td>
<td>2.030653513</td>
</tr>
<tr>
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<td>AE</td>
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<td>2.7977889587</td>
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<td>2.7550895638</td>
<td>0.0005441270</td>
<td>2.6582702568</td>
</tr>
<tr>
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<td>AE</td>
<td>0.0002266868</td>
<td>2.9554505249</td>
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<td>2.9906210160</td>
<td>0.0000837968</td>
<td>2.6390477645</td>
</tr>
<tr>
<td>640</td>
<td>AE</td>
<td>0.0000273512</td>
<td>3.0519218754</td>
<td>0.0000097165</td>
<td>3.0891827424</td>
<td>0.0000117979</td>
<td>2.9668914061</td>
</tr>
</tbody>
</table>

Fig. 11. Surface plot for Example 5.7 at time \( t = \frac{1.5}{\pi} \) with uniform mesh \( \Delta x = \Delta y = \frac{1}{10} \) using dimension-by-dimension approach.

with matching accuracy. Local AE schemes are made possible by choosing the scale parameter \( \epsilon \) to reflect the local distribution of nonlinear waves. The AE schemes have the advantage of easy formulation and implementation and efficient computation of the solution. For the first order local AE scheme and the second order local AE scheme with limiters, we have proved the desired maximum-satisfying property. Numerical tests for both one- and two-dimensional HJ equations presented in this work have demonstrated the high order accuracy and capacity of the AE schemes.

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REFERENCES


