Problem 2. Ralph inherits $2^{25}$ dollars from a long lost relative. He decides to use to gamble in a coin flipping game. The coin is fair, so the probability that the coin will land heads is $1/2$ and the probability that it comes up tails is $1/2$. Ralph bets half the money in his possession with each toss of the coin. If the coin comes up tails, Ralph loses the money bet. If it comes up heads, he wins the amount bet. After 25 tosses, Ralph is ahead. What is the minimal number of games he could have won and what is his minimal possible profit (given that he is ahead) at this time?

Solution. Ralph must have won at least 16 games and must have a profit of at least $9,492,289$.

If Ralph loses then the amount of money he has at the time is multiplied by $\frac{1}{2}$. If he wins, the amount of money he has on hand is multiplied by $\frac{3}{2}$. Thus if Ralph wins $w$ of the bets and loses the other $25-w$ bets, then at the end of the 25 games the number of dollars he has is

$$2^{25} \left(\frac{3}{2}\right)^w \left(\frac{1}{2}\right)^{25-w} = 3^w.$$ 

Note that this is independent of the order of the wins and losses. Ralph comes out ahead if $3^w > 2^{25}$. This is the case if

$$w > \frac{25 \ln 2}{\ln 3} \approx 15.77.$$ 

Thus to come out ahead, Ralph must win at least 16 games. If he does win 16 games then his profit is

$$2^{25} \left(\frac{3}{2}\right)^{16} \left(\frac{1}{2}\right)^{25-16} - 2^{25} = 9,492,289.$$