Problem 12. A 10-digit number

\[ I = d_0d_1d_2d_3d_4d_5d_6d_7d_8d_9 \]

is said to be introspective if for each \( k, 0 \leq k \leq 9 \), \( d_k \) is equal to the number of times that the digit \( k \) appears in \( I \). For example, if \( d_5 = 3 \), then there would be three appearances of the digit 5 in \( I \). Find all ten-digit introspective numbers.

Solution. There is only one introspective ten-digit number: 6210001000. Because \( I \) has ten digits and because \( d_k \) is equal to the number of appearances of the digit \( k \), it must be the case that

\[ d_0 + d_1 + d_2 + d_3 + d_4 + d_5 + d_6 + d_7 + d_8 + d_9 = 10. \tag{1} \]

Furthermore, \( k \cdot d_k \) is the sum of the digits of value \( k \) so is must also be the case that

\[ 0d_0 + 1d_1 + 2d_2 + 3d_3 + 4d_4 + 5d_5 + 6d_6 + 7d_7 + 8d_8 + 9d_9 = 10. \tag{2} \]

From (2) it follows that at most four of \( d_1, d_2, \ldots, d_9 \) can be nonzero, so \( d_0 \geq 6 \). Thus \( d_{d_0} \geq 1 \) so \( d_0(d_{d_0}) \geq d_0 \geq 6 \), and it then follows from (2) that

\[ d_{d_0 + 1} = d_{d_0 + 2} = \cdots = d_9 = 0 \quad \text{and that} \quad d_{d_0} = 1. \tag{3} \]

Subtracting (1) from (2), then solving for \( d_0 \) shows that

\[ d_0 = 1d_2 + 2d_3 + 3d_4 + 4d_5 + 5d_6 + 6d_7 + 7d_8 + 8d_9 = \sum_{k=2}^{9} (k-1)d_k = \sum_{k=2}^{d_0} (k-1)d_k. \]

Using (3) we may write

\[ d_0 = \sum_{k=2}^{d_0-1} (k-1)d_k + (d_0-1) \quad \text{and it follows that} \quad \sum_{k=2}^{d_0-1} (k-1)d_k = 1. \]
This can only be the case if $d_2 = 1$ and $d_3 = d_4 = \cdots = d_{d_0-1} = 0$. Then from (1),

$$d_1 = 10 - d_0 - d_2 - d_3 - \cdots - d_9 = 10 - d_0 - d_0 - d_{d_0} = 8 - d_0.$$ 

Thus we have $6 \leq d_0 \leq 8$ and we have determined all possible values of all digits for the three cases $d_0 = 6, 7, 8$.

If $d_0 = 6$, then $d_1 = 8 - 6 = 2$, $d_2 = 1$, $d_6 = 1$ and all other digits are 0. The resulting number is 6210001000. It is easily checked that this is introspective.

If $d_0 = 7$, then $d_1 = 8 - 7 = 1$, $d_2 = 1$, $d_7 = 1$ and all other digits are 0. The resulting number is 7110000100 but is clearly not introspective.

Finally, if $d_0 = 8$, the $d_1 = 8 - 8 = 0$, $d_2 = 1$, $d_8 = 1$ and all other digits are 0. The resulting number 8010000010 is not introspective.