Problem 1. To celebrate the new year, Emily wrote down a large number by writing the integers 1, 2, 3, ..., 2016 one after another to produce the number

\[ N = 1234 \ldots 201420152016. \]

For full credit answer, with complete explanations, the following questions

a. How many digits are in the number \( N \)?

b. Reading from the left, what is the 2016\textsuperscript{th} digit of \( N \)?

c. What is the remainder when \( N \) is divided by 9?

Solution.

a. The 9 integers 1-9 contribute 9 digits.

The 90 integers 10-99 contribute \( 2 \times 90 = 180 \) digits.

The 900 integers 100-999 contribute \( 3 \times 900 = 2700 \) digits.

The 1017 integers 1000-2016 contribute \( 4 \times 1017 = 4068 \) digits.

Summing these we see that \( N \) is \( 9 + 180 + 2700 + 4068 = 6957 \) digits.

b. The 2016-th digit from the left comes from one of the three digit number contributions. The one and two digit integers contribute 189 digits to \( N \), Because

\[ \frac{2016 - 189}{3} = 609, \]

the 2016-th digit will be the third digit of the 609-th three digit number. Starting from 100, the 609-th three digit number is 708. The third digit in this number is 8.

c. Given and integer, \( M \) partition the string of digits to get several smaller numbers.

Let \( S \) be the sum of the smaller numbers. Then \( M \) and \( S \) have the same remainder when divided by 9. For example, if \( M = 23374199 \) we can cut \( M \) into the three numbers 23,374, 199. The sum of these is \( S = 596 \). It is easy to check that \( M \) and \( S \) both leave a remainder of 2 when divided by 9. This on division by 9, \( N \) will have the same remainder as

\[ S = 1 + 2 + \cdots + 2016 = \frac{2016(2016 + 1)}{2} = 2003136. \]

Because \( S \) leaves remainder 0 when divided by 9, this is also the remainder when \( N \) is divided by 9.