Problem 4. Let \( x, y, z, a, b, c \) be integers that satisfy
\[
x^2 + y^2 = a^2, \quad y^2 + z^2 = b^2, \quad \text{and} \quad z^2 + x^2 = c^2.
\]
Prove that \( xyz \) is a multiple of 55.

Solution. We prove that one of \( x, y, z \) is a multiple of 5 and that one of these three numbers is a multiple of 11.

a) If one of \( x, y, z \) is a multiple of 5, then so is the product \( xyz \). If none of \( x, y, z \) is a multiple of 5, then each of \( x^2, y^2, z^2 \) is congruent to 1 or \(-1\) modulo 5. If two of the three are congruent to 1 modulo 5, then their sum is congruent to 2 modulo five, so the sum cannot be a perfect square. Similar reasoning shows that two of the three cannot be congruent to \(-1\) modulo 5. Thus it is not possible that all three sums are perfect squares unless at least one of \( x, y, z \) is a multiple of 5.

b) The argument to see that at least one of \( x, y, z \) is a multiple of 11 is similar. If none of the three is a multiple of 11, then each of \( x^2, y^2, z^2 \) is congruent to an element of the set \( R = \{-2, 1, 3, 4, 5\} \) modulo 11. If
\[
x^2 \equiv y^2 \equiv r \in \{-2, 1, 3, 4, 5\},
\]
then, because \( 2r \equiv 0 \pmod{11} \notin R \), \( x^2 + y^2 \) cannot be a perfect square. Thus we need only consider the case in which \( x^2, y^2, z^2 \) are congruent to three distinct elements of \( R \). Without loss of generality, we may assume
\[
x^2 \equiv r \pmod{11}, \quad x^2 \equiv s \pmod{11}, \quad x^2 \equiv t \pmod{11},
\]
with \( r, s, t \in R \) and \( r < s < t \). Because \( a^2 = x^2 + y^2 \equiv r + s \pmod{11} \) and \( b^2 = y^2 + z^2 \equiv s + t \pmod{11} \), we must also have \( r + s, s + t \) congruent (modulo 11) to elements in \( R \). It easy to check that so such ordered triple \((r, s, t)\) of elements in \( R \) satisfies these conditions. Therefore at least one of \( x, y, z \) must be a multiple of 11.

Because at least one of \( x, y, z \) is a multiple of 5, at least one os a multiple of 11, and 5 and 11 are relatively prime, it follows that \( xyz \) is a multiple of 55.

Such triples \((x, y, z)\) do exist, for example \((44, 117, 240)\).