Problem 3. Several tennis players compete in a tournament in which each player plays against every other player exactly once. Show that at any given time during the tournament there are two players who have played the same number of games.

Solution. Assume the tournament has $n \geq 2$ players and that we are at some point in the tournament. We consider two cases:

a) If one of the players has played $n-1$ games (and so has played every other player in the tournament) then no player has played 0 games. Thus the number of games played by each player is an element of the set $\{1, 2, \ldots, n-1\}$. Because this set has $n-1$ elements and there are $n$ players, some pair of players must have played the same number of games. (This is an application of the pigeon-hole principle.)

b) If none of the players have played $n-1$ games, then the number of games played by each player is an element of the $(n-1)$-element set $\{0, 1, \ldots, n-2\}$. Once again the pigeon-hole principle implies that there are two of the $n$ players who have played the same number of games.

Because one of these two cases must be true at every point of the tournament, we conclude that at any time there are a pair of players who have played the same number of games.