Problem 2. Let \( I_{2014} \) be the 2014-digit number all of whose digits are equal to 1, and \( I_{1007} \) be the 1007-digit number all of whose digits are equal to 1. Prove, without the aid of a calculator or computer, that

\[
I_{2014} + 4I_{1007} + 1
\]

is a perfect square.

Solution. If \( I_n \) is the \( n \)-digit number all of whose digits are equal to 1, then

\[
I_n = \sum_{k=0}^{n-1} 10^k = \frac{10^n - 1}{9}.
\]

Thus

\[
I_{2n} + 4I_n + 1 = \frac{10^{2n} - 1}{9} + 4 \cdot \frac{10^n - 1}{9} + 1 = \frac{10^{2n} + 4 \cdot 10^n + 4}{9} = \left( \frac{10^n + 2}{3} \right)^2.
\]

Because \( 10 \equiv 1 \pmod{3} \), it follows that \( 10^n + 2 \equiv 0 \pmod{3} \), so \( \frac{10^n + 2}{3} \) is an integer. Thus \( I_{2n} + 4I_n + 1 \) is the square of an integer.