Problem 15. Let each of $a_0, a_1, \ldots, a_{n-1}$ be equal to $\pm 1$. Prove that

$$2 \sin \left( \left( a_0 + \frac{a_0a_1}{2} + \cdots + \frac{a_0a_1 \cdots a_{n-1}}{2^{n-1}} \right) \frac{\pi}{4} \right) = a_0 \sqrt{2 + a_1 \sqrt{2 + a_2 \sqrt{2 + \cdots + a_{n-1} \sqrt{2}}}}.$$

Solution. We prove the result using mathematical induction. The base case, $n = 1$ is true because for $a_0 = \pm 1$ (and because sine is an odd function)

$$2 \sin \left( a_0 \frac{\pi}{4} \right) = 2 a_0 \sin \frac{\pi}{4} = a_0 \sqrt{2}.$$

Now assume that the result is true for any choice of $a_0, a_1, \ldots, a_{n-2} \in \{-1, 1\}$. Now assume that $a_{n-1} \in \{-1, 1\}$ is given and let

$$\theta = \left( a_1 + \frac{a_1a_2}{2} + \cdots + \frac{a_1a_2 \cdots a_{n-1}}{2^{n-2}} \right) \frac{\pi}{4}.$$

Because sine is an odd function we have

$$2 \sin \left( \left( a_0 + \frac{a_0a_1}{2} + \cdots + \frac{a_0a_1 \cdots a_{n-1}}{2^{n-1}} \right) \frac{\pi}{4} \right) = 2 a_0 \sin \left( \frac{\pi}{4} + \frac{1}{2} \theta \right)$$

$$= 2 a_0 \sqrt{\sin^2 \left( \frac{\pi}{4} + \frac{1}{2} \theta \right)} = 2 a_0 \sqrt{1 - \cos \left( \frac{\pi}{2} + \theta \right)}$$

$$= 2 a_0 \sqrt{\frac{1 + \sin \theta}{2}} = a_0 \sqrt{2 + 2 \sin \theta}$$

$$= a_0 \sqrt{2 + 2 \sin \left( \left( a_1 + \frac{a_1a_2}{2} + \cdots + \frac{a_1a_2 \cdots a_{n-1}}{2^{n-2}} \right) \frac{\pi}{4} \right)}.$$

where the last line follows from the induction hypothesis.

The formula presented here appeared in the paper “A Class of Continued Radicals” by Costas J. Efthimiou in the May 2013 issue of *The American Mathematical Monthly*. 

1