Problem 11. A King takes a tour of an $8 \times 8$ chessboard. Obeying the rules for how a King can move, at each step he moves from the presently occupied square to one of eight adjacent squares (four of these eight squares are “diagonal” and share only a corner with the occupied square, and four of the squares share an edge with the occupied square. The King never moves off of the board.) Starting from a square $S$, the King makes 64 moves, visiting each square exactly once and finishing again in square $S$. Prove that the King made an even number of diagonal moves.

Solution. In touring the board, the King makes 64 moves; for the last move the King returns to its starting square. Consider the color changes the King makes in moving on the board: with each vertical or horizontal move the King moves to a square of a different color; with each diagonal move, the King moves from a square to a square of the same color. Because the King starts and ends on the same square, there must be an even number of color changes, so the number moves that are horizontal or vertical moves must be even. Because the total number of moves is even, there must also have been and even number of diagonal moves.