Problem 12. Find positive integers \(a\) and \(b\) so that \(a + b\) is as small as possible and

\[ a \mid b^2, \quad b^2 \mid a^3, \quad a^3 \mid b^4, \quad b^4 \mid a^5 \quad \text{but} \quad a^5 \nmid b^6. \]

(For positive integers \(m\) and \(n\), \(m \mid n\) means that \(m\) is a factor of \(n\).)

Solution 12. It is clear that \(a\) and \(b\) must have the same prime factors. Furthermore if

\[ a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n} \quad \text{and} \quad b = p_1^{\beta_1} p_2^{\beta_2} \cdots p_n^{\beta_n}, \]

where \(p_1, \ldots, p_n\) are distinct primes, then for each \(k\),

\[ p_k^{\alpha_k} \mid (p_k^{\beta_k})^2, \quad (p_k^{\beta_k})^2 \mid (p_k^{\alpha_k})^3, \quad (p_k^{\alpha_k})^3 \mid (p_k^{\beta_k})^4, \quad \text{and} \quad (p_k^{\beta_k})^4 \mid (p_k^{\alpha_k})^5. \]

Furthermore, to satisfy the “nondivisibility” condition, there must be a \(k_0\) so that

\[ (p_k^{\alpha_{k_0}})^5 \nmid (p_k^{\beta_{k_0}})^6. \]

It follows that the example of such integers \(a\) and \(b\) for which \(a\) and \(b\) are minimal will occur when \(a = 2^\alpha\) and \(b = 2^\beta\) for some nonnegative integers \(\alpha\) and \(\beta\). We must then have

\[ 2^\alpha \mid 2^{2\beta}, \quad 2^{2\beta} \mid 2^{3\alpha}, \quad 2^{3\alpha} \mid 2^{4\beta}, \quad 2^{4\beta} \mid 2^{5\alpha}, \quad \text{and} \quad 2^{5\alpha} \nmid 2^{6\beta}. \]

Hence

\[ \alpha \leq 2\beta \leq 3\alpha \leq 4\beta \leq 5\alpha, \quad \text{and} \quad 5\alpha > 6\beta. \]

A little algebra shows that integers \(\alpha\) and \(\beta\) satisfy these inequalities if and only if

\[ \frac{3}{4} \leq \frac{\beta}{\alpha} < \frac{5}{6}. \]

The choice \(\alpha = 4, \beta = 3\), gives the smallest possible exponents and the solution \(a = 16, b = 8\) with the minimal value of \(a + b\).