Problem 10. Prove that the polynomial $x^{2n} + x^n + 1$ is divisible by $x^2 + x + 1$ if and only if $n$ is not a multiple of 3.

Solution 10. The solutions to the equation $x^2 + x + 1 = 0$ are

$$\omega = \frac{-1 + i\sqrt{3}}{2} \quad \text{and} \quad \overline{\omega} = \frac{-1 - i\sqrt{3}}{2}.$$ 

Note too that $\omega^3 = \overline{\omega}^3 = 1$, and that $\omega^2 + \omega + 1 = \overline{\omega}^2 + \overline{\omega} + 1 = 0$. Now let $n = 3k$ be a multiple of 3. Then

$$\omega^{2n} + \omega^n + 1 = (\omega^3)^{2k} + (\omega^3)^k + 1 = 1 + 1 + 1 = 3.$$ 

Because $\omega$ is not a zero of $x^{2n} + x^n + 1$, it follows that $x - \omega$ is not a factor of this polynomial. Thus $x^2 + x + 1$ is not a factor of $x^{2n} + x^n + 1$ if $n$ is a multiple of 3.

If $n$ is not a multiple of 3, then $n = 2k + 1$ or $n = 2k + 2$ for some nonnegative integer $k$. We consider the case $n = 2k + 1$; the other case is handled similarly. We then have

$$\omega^{2n} + \omega^n + 1 = (\omega^3)^{2k} \omega^2 + (\omega^3)^k \omega + 1 = \omega^2 + \omega + 1 = 0.$$ 

Because $\omega$ is a zero of $x^{2n} + x + 1$, it follows that $x - \omega$ is a factor of $x^{2n} + x + 1$. A similar argument shows that $x - \overline{\omega}$ is also a factor of $x^{2n} + x^n + 1$. Because $x - \omega$ and $x - \overline{\omega}$ have no common factor, it follows that

$$(x - \omega)(x - \overline{\omega}) = x^2 + x + 1$$

is a factor of $x^{2n} + x^n + 1$.  
