Problem 6. MiDiOwA Airlines (slogan: We have our ups and downs), services several mid-Iowa towns, and uses the following rule to determine its air routes: If \( A \) and \( B \) are two different towns, then there is an air route connecting \( A \) with \( B \) if and only if there is no town closer to \( A \) than \( B \) or there is no town closer to \( B \) than \( A \). No other routes are created. If each distance between a pair of towns is different from the distance between every other pair of towns, prove that no town can be connected by a direct flight to more than five other towns.

Solution. Suppose there are six cities \( A_1, A_2, \ldots, A_6 \) connected by direct flights to a city \( P \). Then there is a pair \( i, j \) so that the measure of (acute) \( \angle A_i PA_j \) is at most \( 60^\circ \). Without loss of generality let \( \angle A_1 CA_2 \) have measure less than \( 60^\circ \). Because the distance between each pair of towns is different than the distance between any other pair, \( \triangle A_1 PA_2 \) is scalene, and the largest angle has measure greater than \( 60^\circ \). We may assume, again without loss of generality, that this is \( \angle A_2 A_1 P \). Thus

\[
P A_2 > P A_1 \quad \text{and} \quad A_2 P > A_2 A_1.
\]

Thus \( A_2 \) is not the city closest to \( P \), and \( P \) is not the city closest to \( A_2 \). This contradicts the assumption that there is a direct flight between \( P \) and \( A_2 \).