Problem 15. The lengths of the three sides of a triangle form a three term arithmetic progression and the lengths of the three altitudes of the triangle also form a three term arithmetic progression. Prove that the triangle is equilateral.

Solution. Let the sides of the triangle have lengths $s, s + a, s + 2a$ in increasing order, and let the altitudes have lengths $h, h + b, h + 2b$ in increasing order. Then

$$2A = s(h + 2b) = (s + a)(h + b) = (s + 2a)h,$$

where $A$ is the area of the triangle. This implies that

$$0 = 2(s + a)(h + b) - s(h + 2b) - (s + 2a)h = 2ab.$$

It follows that at least one of $a$ or $b$ is equal to 0, and then from (1) it follows that both must be equal to 0. Therefore the sides of the triangle all have length $s$, so the triangle is equilateral.