Problem 14. Prove that the positive number $x$ is rational if and only if the sequence 

$$x, x + 1, x + 2, x + 3, \ldots$$

contains three terms that form a geometric progression.

Solution. First assume that the sequence has three terms that form a geometric sequence. Without loss of generality we may assume that the first term of this geometric sequence is $x$. (If $x + N$ is the first term of the sequence, delete the terms $x, x + 1, \ldots, x + (N - 1)$ and relabel $x + N$ as $x$ and, for $M > N$, $x + M$ as $x + M - N$. Note that $x$ is rational if and only if $x + M$ is rational.) Thus there are positive integers $m, n$ with $m < n$ and

$$\frac{x}{x + m} = \frac{x + m}{x + n}.$$ 

This leads to the equation $(n - 2m)x = m^2$. Because $m$ and $n$ are positive integers, it cannot be the case that $n = 2m$. Thus

$$x = \frac{m^2}{n - 2m}, \quad (1)$$

showing that $x$ is rational.

Conversely, if $x = \frac{p}{q}$ is rational, where $p$ and $q$ are positive integers, then it is easy to verify that

$$\frac{p}{q}, \frac{p}{q} + p, \frac{p}{q} + (pq + 2p)$$

are in arithmetic progression. To produce this sequence, we simply need to find $m$ and $n$ so that (1) is true, that is

$$\frac{p}{q} = \frac{m^2}{n - 2m}.$$ 

However, this is equivalent to

$$\frac{p^2}{pq} = \frac{m^2}{n - 2m},$$

and then the choices $m = p, n = pq + 2p$ follow immediately.