Problem 7. Cy (of Cyclone fame) has a drawer containing $S$ socks some of which are cardinal and the rest of which are gold. If he reaches into the drawer and selects two of the socks at random, the probability that the two socks are the same color is exactly $\frac{1}{2}$. Find all possible values for $S$.

Solution. The number of socks can be any square positive integer greater than or equal to 4.

Let $S$ be the number of socks and $c$ the number of cardinal socks (so there are $S - c$ gold socks.) The probability of drawing two socks of the same color is the probability of drawing two cardinal socks plus the probability of drawing two gold socks. Thus we must have

$$\frac{\binom{c}{2}}{\binom{S}{2}} + \frac{\binom{S-c}{2}}{\binom{S}{2}} = \frac{1}{2}.$$

This reduces to

$$2c(c-1) + 2(S-c)(S-c-1) = S(S-1)$$

and then to

$$4c^2 - (4S)c + (S^2 - S) = 0.$$

Solving for $c$ we find

$$c = \frac{4S \pm \sqrt{(4S)^2 - 4(4)(S^2 - S)}}{2 \cdot 4} = \frac{S \pm \sqrt{S}}{2}.$$

Because $c$ must be a positive integer, $S$ must be a positive square, and it is clear that $S$ must be greater than 1. Thus the set of possible values for $S$ is $\{n^2 : n \geq 2\}$. For $S = n^2$ there are

$$\frac{n^2 - n}{2}$$

socks of one color and $\frac{n^2 + n}{2}$ of the other.

Thus the numbers of cardinal and gold socks are consecutive triangular numbers!

Note. A number of students found a useful quadratic and were able, through examples, to guess that $S$ must be a square. However, it was necessary to prove that all squares greater than 1 are possibilities.