Problem 11. In a convex quadrilateral, the two diagonals cut the interior into four triangles, each with integer area. Prove that the product of these areas is a perfect square.

Solution. Let the vertices of the quadrilateral be $A, B, C, D$ and denote the intersection point of the diagonals by $P$, as seen in the Figure below. Because $\triangle CDP$ and $\triangle DAP$ have the same altitude from $D$, this altitude $h_D$ is given by

$$\frac{\text{Area}(CDP)}{CP} = h_D = \frac{\text{Area}(DAP)}{AP}.$$ 

By similar reasoning,

$$\frac{\text{Area}(BCP)}{CP} = h_B = \frac{\text{Area}(ABP)}{AP}.$$ 

Therefore

$$\text{Area}(CDP) = h_D(CP), \quad \text{Area}(ABP) = h_B(AP), \quad \text{Area}(DAP) = h_D(AP)$$

and

$$\text{Area}(BCP) = h_B(CP).$$

Thus, taking the products of the areas of pairs of triangles that intersect only at $P$ we see

$$(\text{Area}(CDP))(\text{Area}(ABP)) = h_B h_D(AP)(CP) = (\text{Area}(DAP))(\text{Area}(BCP)).$$

Because each of the areas is an integer, it follows that

$$(\text{Area}(CDP))(\text{Area}(ABP))(\text{Area}(DAP))(\text{Area}(BCP)) = (\text{Area}(DAP))(\text{Area}(BCP))^2$$

is a perfect square.

![Figure 1: Each triangle has integral area.](image-url)