Problem 12. An ellipse in the plane has foci at (20, 19) and (55, 59) and is tangent to the y-axis. What is the length of the major axis of the ellipse?

Solution. Let $F = (55, 59)$, $G = (20, 19)$ and let $2a$ be the length of the major axis of the ellipse. Then the ellipse with foci $F$ and $G$ major axis of length $2a$ is the the set of all points $P$ with

$$|PG| + |PF| = 2a.$$ 

Let $T$ be the point at which the ellipse is tangent to the y-axis. By the well known reflection property of the ellipse, a ray of light from one focus will reflect off a mirrored inner surface of the ellipse and pass through the other focus. If the ray starts at $G$ and reflects at $T$, then the incident and reflected rays make the same angle $\phi$ with the y-axis. See the figure below. Thus the distance

$$2a = |GT| + |TF| = |GT| + |TF'| = |GF'|,$$

where $F' = (-55, 59)$ is the reflection of $F$ in the y-axis. Thus the length of the major axis is

$$2a = |GF'| = \sqrt{(20 - (-55))^2 + (19 - 59)^2} = 85.$$