Problem 9. Prove that

\[ 3^{2013} \] is a factor of \( 2^{(3^{2013})} + 1 \).

Solution. We prove by mathematical induction that for any nonnegative integer \( m \), we have

\[ 3^m | (2^{(3^m}) + 1). \]  \hspace{1cm} (1)

(For integers \( a \) and \( b \) the notation \( a|b \) means that \( a \) is a factor of \( b \).)

It is easy to verify that (1) is true for \( m = 0 \) and for \( m = 1 \). Now assume that (1) holds for an integer \( m \geq 1 \). Then there is a positive integer \( k \) for which

\[ 2^{(3^m)} + 1 = k(3^m) \] so that \( 2^{(3^m)} = k(3^m) - 1 \).

Therefore

\[ 2^{(3^{m+1})} = \left( 2^{(3^m)} \right)^3 = (k(3^m) - 1)^3 = k^3 3^{3m} - 3k^2 3^{2m} + 3k 3^m - 1. \]

From this we have

\[ 2^{(3^{m+1})} + 1 = 3^{m+1} (k^3 3^{2m-1} - k^2 3^m + k), \]

showing that \( 3^{m+1} \) is a factor of \( 2^{(3^{m+1})} + 1 \). This completes the proof that (1) is true for all nonnegative integers \( m \), and in particular for the value \( m = 2013 \).