Problem 6. Let $S$ be the set of all positive rational numbers in lowest terms and with denominator less than 100. Find the element in $S$ that is closest to, but not equal to, $\frac{5}{7}$.

(This must be done without the use of calculator or computer. Correct solutions will have the numerical (fraction) answer and complete supporting work.)

Solution. Let $\frac{p}{q} \neq \frac{5}{7}$ be a number in $S$. Then

$$\left|\frac{p}{q} - \frac{5}{7}\right| = \frac{|7p - 5q|}{7q}.$$

We first find the fraction $\frac{p}{q}$ for $|7p - 5q| = 1$ and $7q$ is as large as possible. First assume $7p - 5q = 1$. One solution to this equation is $(p, q) = (3, 4)$. All other solutions with both $p$ and $q$ positive are given by

$$(p, q) = (3 + 5k, 4 + 7k), \quad k = 0, 1, 2, \ldots.$$  

We want $q = 4 + 7k \leq 99$ and as large as possible, so take $k = 13$. This results in $q = 95$, $p = 68$ and the fraction $\frac{68}{95} \in S$.

Next assume $7p - 5q = -1$, for which the positive integer solutions are

$$(p, q) = (2 + 5k, 3 + 7k), \quad k = 0, 1, 2, \ldots.$$  

For $q = 3 + 7k \leq 99$ as large as possible, take $k = 13$ leading to $\frac{p}{q} = \frac{67}{94} \in S$. Because

$$\frac{68}{95} - \frac{5}{7} = \frac{1}{7 \cdot 95} \quad \text{and} \quad \frac{67}{94} - \frac{p}{q} = -\frac{1}{7 \cdot 94},$$

we conclude that $\frac{68}{95}$ is closer to $\frac{5}{7}$ than is $\frac{67}{94}$.

Finally, if $\frac{p}{q} \in S$ and

$$\left|\frac{p}{q} - \frac{5}{7}\right| = \frac{|7p - 5q|}{7q}$$

has numerator greater than or equal to 2, then

$$\frac{|7p - 5q|}{7q} \geq \frac{2}{7 \cdot 99} > \frac{1}{7 \cdot 95}.$$  

It follows that $\frac{68}{95}$ is the element of $S$ that is closest to, but not equal to, $\frac{5}{7}$.  

1
The solution can also be addressed through Farey fractions. The Farey construction starts with the fractions \( \frac{0}{1} \) and \( \frac{1}{1} \), as shown in the top line of the display below. The rows are constructed iteratively: once row \( n \) is constructed, obtain row \( n + 1 \) by inserting, between each pair of adjacent fractions in row \( n \), the mediant* of the two fractions. The first five rows of the construction are shown below.

\[
\begin{array}{cccc}
0 & 1 & 1 & 1 \\
1 & 2 & 1 & 1 \\
1 & 2 & 3 & 1 \\
2 & 3 & 4 & 1 \\
\end{array}
\]

It can be shown that every rational number between 0 and 1 eventually appears in the construction and appears in lowest terms. In addition, the numbers appear in each row in increasing order. As seen above, the fraction \( \frac{5}{7} \) first appears in the fifth row. It is not hard to show that in row 18, \( \frac{5}{7} \) is flanked by \( \frac{67}{94} \) and \( \frac{68}{95} \), and all numbers flanking \( \frac{5}{7} \) in subsequent rows have denominators greater than 99. Thus the number in \( S \) closest to \( \frac{5}{7} \) is either \( \frac{67}{94} \) or \( \frac{68}{95} \).

* The mediant of two fractions \( \frac{a}{b} \) and \( \frac{c}{d} \) is \( \frac{a+c}{b+d} \).