**Problem 10.** In triangle $ABC$, point $A'$ is on $BC$, $B'$ is on $CA$ and $C'$ is on $AB$ as shown below. Given that the circle through points $A$, $B'$, $C'$ and the circle through $B$, $C'$, $A'$ intersect at a point $P$ in the interior of the triangle, prove that the circle through $C$, $A'$, $B'$ also passes through $P$.

![Diagram of triangle ABC with points A', B', C' and circle passing through them](image)

**Solution.** Draw the segments from $P$ to each of $A'$, $B'$, and $C'$. Quadrilaterals $AC'B'P$ and $B'A'C'$ are both cyclic quadrilaterals (i.e., for each quadrilateral, the four vertices lie on a circle.) We use the following well known theorem:

Let $WXYZ$ be a quadrilateral. Then $WXYZ$ is cyclic if and only if

$$\angle X + \angle Z = 180^\circ.$$  

Thus in quadrilateral $AC'PA'$ we have $\angle PB'A + \angle PC'A = 180^\circ$ and in $BA'PC'$, $\angle PC'B + \angle PA'B = 180^\circ$. Also note that $\angle PC'A + \angle PC'B = 180^\circ$. Using this we prove that in quadrilateral $CB'PA'$ we have $\angle PB'C + \angle PA'C' = 180^\circ$, which will imply that $CB'PA'$ is cyclic. We have

$$\angle PB'C = 180^\circ - \angle PB'A = 180^\circ - (180^\circ - \angle PC'A)$$
$$= \angle PC'A = 180^\circ - \angle PC'B = 180^\circ - (180^\circ - \angle PA'B)$$
$$= \angle PA'B = 180^\circ - \angle PA'C.$$

This completes the proof.