Problem 10. Given a positive integer $n$, let $\rho(n)$ be the sum of all of the positive integers that are less than $n$ and relatively prime to $n$. Are there any positive integers $n$ such that $\rho(n) = 3n$? Either find all such positive integers or prove no such integers exist.

Solution. We show that the only such integers are 7, 9, 14, 18.

First note that for integer $k$ with $1 \leq k \leq n$, $k$ is relatively prime to $n$ if and only if $n - k$ is also relatively prime to $n$. Thus

$$\rho(n) = \frac{\phi(n)}{2}n,$$

where $\phi(n)$ is Euler’s $\phi$ function and gives the number of positive integers less than or equal to $n$ and relatively prime to $n$. Thus $\rho(n) = 3n$ if and only if $\phi(n) = 6$.

Let positive integer $n = p_1^{r_1}p_2^{r_2}\cdots p_m^{r_m}$ where $p_1, p_2, \ldots, p_m$ are distinct primes and $r_1, r_2, \ldots, r_m$ are positive integers. It is well known that

$$\phi(n) = p_1^{r_1-1}(p_1 - 1)p_2^{r_2-1}(p_2 - 1)\cdots p_m^{r_m-1}(p_m - 1).$$

It follows that if $\phi(n) = 6$, then $n$ can have no prime factor $p_k \geq 8$, so the only possible prime factors of $n$ are 2, 3, 5, 7. If $p_k = 7$ and $r_k \geq 2$, then $\phi(n) \geq 7^{2-1}(7 - 1) > 6$, so the exponent on a prime factor of 7 can only be 0 or 1. If $p_k = 5$, then $\phi(n)$ has a factor of $5 - 1 = 4$, so no multiple of 5 is a candidate for $n$. If $p_k = 3$ and $r_k \geq 3$, then $\phi(n) \geq 3^{3-1}(3 - 1) > 6$, so the exponent on a prime factor of 3 can only be 0, 1, or 2. By similar reasoning, the exponent on a prime factor of 2 can only be 0, 1, 2, or 3.

Thus, the only possible values of $n$ are $n = 2^a3^b7^c$ where $a = 0, 1, 2$, $b = 0, 1, 2$, and $c = 0, 1$. Because $\phi(n) = 6$ means $n \geq 7$, the possible values of $n$ are

$$7, 8, 9, 12, 14, 18, 21, 24, 28, 36, 42, 56, 63, 72, 84, 126, 16, 252, 504.$$ 

We can eliminate some of these by noting that $n$ cannot be a multiple of both 3 and 7 because then $\phi(n) > 6$. Eliminating multiples of 21 leaves the list

$$7, 8, 9, 12, 14, 18, 24, 28, 36, 56, 72.$$ 

A quick check shows that $\phi(n) = 6$ only for $n = 7, 9, 14, 18$. 

1