Problem 5. A total of 100 mages and necromancers are seated around a round table. Necromancers always lie, and mages always speak the truth, except when they are joking. Everybody declares that they are sitting between a mage and a necromancer, but it turns out that two of the mages are joking. How many necromancers are there at the table?

We know that there are at least two Mages (M) because we were told that two of them were joking. Also, because every Necromancer (N) lies, every Necromancer must be between two other Necromancers or between two Mages. We first note that it is impossible for one Necromancer to be between two others. To prove this, assume that there were three Necromancers in a row: \( N N N \). From here, continue around the circle (say in the clockwise direction) until we meet a Mage:

\[
N N N \ldots N N M.
\]

Then the last Necromancer met is between a Necromancer and a Mage. This is impossible because if this were the case, then the Necromancer must have been telling the truth. Thus every Necromancer sits between two Mages. Hence there can be no more than 50 Necromancers and must be at least 50 Mages. Likewise, all but two of the Mages must sit between a Mage and a Necromancer, so most of the Mages must sit in pairs, with two Mages between a pair of consecutive Necromancers:

\[
\ldots N M M N M M N M M \ldots
\]

We now show by the process of elimination that the number of Necromancers must be 34.

First assume that there are 35 or more Necromancers. Then there are 65 or fewer Mages. We know from the above arguments that we cannot have two Necromancers adjacent. Thus there is at least one Mage between each consecutive pair of Necromancers. Because there are at least 35 “gaps” between adjacent Necromancers, it follows that at least 5 = 2 \cdot 35 − 65 of these gaps that have exactly one Mage. But this means that there are at least 5 joking Mages, because each will be between two Necromancers.

Now assume that there are 33 or fewer Necromancers, so there are 67 or more mages. Again put Mages into the gaps between consecutive Necromancers, again requiring
at least one Mage in each gap. Clearly there can be at most two "gaps" with just one Mage and all others must have at least two Mages.

a. If there are two gaps with just one Mage, then accounting for two Mages in the other gaps brings the count to at most $2 \cdot 31 + 2 = 64$ Mages and the other unaccounted for Mages must be adjacent to a another pair of Mages in some gap. This results in at least three joking Mages.

b. If there is only one gap with just one Mage and all others have at least two, this brings the count to at least $2 \cdot 32 + 1 = 65$ mages, so again there are (at least) 2 remaining Mages must be adjacent to other pairs (or a pair). This again leads to at least 3 joking Mages.

c. If every gap has at least two Mages in it we have accounted for a multiple of three Mages and Necromancer, by counting two of the Mages that follow each Necromancer. Then the remaining number of Mages is 1 or 4 or 7, etc. Each of these remaining Mages must be adjacent to some pair of Mages, and in this way we can account for just 1 or at least 4 joking Mages,

This argument shows that if a configuration as described is possible, then it must have 34 Necromancers. Such a configuration is possible: considering the 34 gaps between adjacent Necromancers, fill 32 of the gaps with 2 Mages and the remaining two with 1 Mage each.