Problem 7. Is there a positive integer $n$ such that $2^n$ has leading digits 77? Justify your answer.

Solution. There are infinitely many powers of 2 that start with the digits 77. The smallest such power is $2^{86}$ and was found by all solvers. This is a special case of a more general result:

*Given any digit sequence, there are infinitely many powers of two that begin with that digit sequence.*

We prove this for the digit sequence 77, but the argument generalizes in an obvious way.

We seek positive integers $m$ and $n$ with

$$77 \cdot 10^m \leq 2^n < 78 \cdot 10^m.$$  

Thus

$$m + \log 77 \leq n \log 2 < m + \log 78,$$

where logarithms are in base 10. For real number $x$, let $\{x\} = x - \lfloor x \rfloor$ be the fractional part of $x$. First observe that because $\log 2$ is irrational, the numbers

$$\{n \log 2\}, \quad n = 1, 2, 3, \ldots$$

are distinct. Indeed if there are distinct positive integers $j, k$ with

$$\{j \log 2\} = \{k \log 2\},$$

then $j \log 2 - k \log 2$ is an integer. But this is impossible if $\log 2$ is irrational.

Now let $\epsilon = \log 78 - \log 77$ and cut the interval $[0, 1)$ into pieces of length less than $\epsilon$. Suppose there are $N$ such pieces and consider the numbers

$$\{\log 2\}, \{2 \log 2\}, \ldots, \{N \log 2\}, \{(N + 1) \log 2\}.$$  

By the Pigeonhole Principle, two of these numbers must be in one of the $N$ intervals. Suppose that these two numbers are

$$\{j \log 2\} \quad \text{and} \quad \{k \log 2\},$$

with $j \lt k$. Let $M = k - j$. Then either

$$\{M \log 2\} = \delta \quad \text{or} \quad \{M \log 2\} = 1 - \delta,$$

where $\delta < \epsilon$. In the first case, when the sequence

$$\{M \log 2\}, \{2M \log 2\}, \{3M \log 2\}, \ldots,$$
is plotted in $[0, 1)$, initially, each point is less than $\epsilon$ to the right of the previous. Because $\log 78 - \log 77 = \epsilon$, there must be integers $N$ and $J$ so that

$$N + \log 77 \leq JM \log 2 < N + \log 78, \quad \text{that is} \quad 77 \cdot 10^N \leq 2^J < 78 \cdot 10^N,$$

and the leading two digits of $2^J$ are 77. A similar argument handles the $\{M \log 2\} = 1 - \delta$ case.