Problem 2. John is on a large, flat (planar) playground. He labels two points, $A$ and $B$ on the playground, then walks in such a way that his distance from point $A$ is always exactly twice his distance from point $B$. Describe John’s walking path and supply work in support of your claim.

John’s path is a circle. In fact, it is one of the Circles of Apollonius associated with $A$ and $B$.

Solution 2A. Put the problem in the coordinate plane with $A = (0, 0)$ and $B = (b, 0)$. A point $P = (x, y)$ is on John’s path if and only if

$$\sqrt{x^2 + y^2} = 2\sqrt{(x-b)^2 + y^2}.$$

Squaring both sides and simplifying we see that $P$ is on the path if and only if

$$3x^2 + 3y^2 - 8bx + 4b^2 = 0.$$ 

Dividing by 3 and completing the square in $x$ this becomes

$$\left(x - \frac{4}{3}b \right)^2 + y^2 = \frac{4}{9}b^2.$$

This shows that John’s path follows the circle of center $\left(\frac{4}{3}b, 0 \right)$ and radius $\frac{2}{3}b$.

Note. Many students arrived at an equation equivalent to the one derived here. However, not all of these students recognized that the graph of the equation is a circle.

Solution 2B. Let $P$ be any point on John’s path and consider triangle $PAB$, with $PA = 2PB$. Now consider the (internal) angle bisector for $\angle P$, and let $R$ be the point in which this bisector intersects $\overline{AB}$. By the angle bisector theorem

$$\frac{AR}{BR} = \frac{AP}{BP} = 2,$$

that is, $R$ (internally) divide segment $AB$ in the ratio 2 : 1. Thus for any $P$ on John’s path, the angle bisector of $\angle APB$ passes through $R$.

A version of the angle bisector theorem is also valid for the external angles of a triangle. In particular, the external bisector at angle $P$ will intersect line $AB$ in a point $S$ with

$$\frac{AS}{BS} = \frac{AP}{BP} = 2.$$
Thus for each point $P$ on John’s path the internal bisector of $\angle P$ passes through $R$ and the external bisector of $\angle P$ passes through $S$. Because these two bisectors are perpendicular at $P$, it follows that for any such point $P$, triangle $RPS$ is a right triangle with hypotenuse $RS$. Hence $P$ lies on the circle with diameter $RS$.

The Circles of Apollonius can also be approached through inversive geometry. Given a circle $\omega$ with center $O$ and radius $r$, let $A$ and $B$ be points on a radial segment with

$$ AO \cdot BO = r^2. $$

Then $A$ and $B$ are said to be *inverses with respect to* $\omega$. The circle $\omega$ is a Circle of Apollonius associated with $A$ and $B$. In particular there is a constant $\lambda$ so that for every point $P$ on $\omega$, we have

$$ \frac{AP}{BP} = \lambda. $$