**Problem 11.** Let $k$ and $m$ be positive integers and let $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_m$ be positive real numbers. Prove that if
\[
\sqrt[n]{a_1} + \sqrt[n]{a_2} + \cdots + \sqrt[n]{a_k} = \sqrt[n]{b_1} + \sqrt[n]{b_2} + \cdots + \sqrt[n]{b_m},
\]
for every positive integer $n$, then
\[
k = m \quad \text{and} \quad a_1a_2\cdots a_k = b_1b_2\cdots b_m.
\]

**Solution.** We apply the two limit results, both valid for any $a > 0$,
\[
\lim_{n \to \infty} \sqrt[n]{a} = 1 \quad \text{and} \quad \lim_{n \to \infty} n(\sqrt[n]{a} - 1) = \ln a. \tag{1}
\]
Both can be proved using L’Hopital’s Rule. Taking the limit as $n$ approaches infinity on both sides of the given equation,
\[
\lim_{n \to \infty} (\sqrt[n]{a_1} + \sqrt[n]{a_2} + \cdots + \sqrt[n]{a_k}) = \lim_{n \to \infty} (\sqrt[n]{b_1} + \sqrt[n]{b_2} + \cdots + \sqrt[n]{b_m}).
\]
Applying the first of the results in (1) leads to $k = m$. Next subtract 1 from each term on each side of the given equation and multiply the result by $n$ to see
\[
n(\sqrt[n]{a_1} - 1) + n(\sqrt[n]{a_2} - 1) + \cdots + n(\sqrt[n]{a_k} - 1) = n(\sqrt[n]{b_1} - 1) + n(\sqrt[n]{b_2} - 1) + \cdots + n(\sqrt[n]{b_k} - 1).
\]
Take the limit as $n$ approaches infinity on both sides and use the second result in (1) to see
\[
\ln a_1 + \ln a_2 + \cdots + \ln a_n = \ln b_1 + \ln b_2 + \cdots + \ln b_k.
\]
thus
\[
\ln(a_1a_2\cdots a_n) = \ln(b_1b_2\cdots b_n),
\]
and it follows that $a_1a_2\cdots a_n = b_1b_2\cdots b_n$. 