Problem 10. For each permutation $a_1, a_2, \ldots, a_{20}$ of the integers $1, 2, \ldots, 20$, form the sum

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{19} - a_{20}|.$$ 

Find the average value of all such sums.

Solution. We solve the problem for the numbers $1, 2, \ldots, 2n$. For the case given in the problem, $n = 10$.

First note that the average value of the sums is just $n$ times the average value of $|a_1 - a_2|$ because the average value of $|a_{2i-1} - a_{2i}|$ is the same for all $i = 1, 2, \ldots, n$. The average value for $|a_1 - a_2|$ for the case $a_1 = k$ is

$$\frac{(k - 1) + (k - 2) + \cdots + 1 + 0 + 1 + \cdots + (2n - k)}{2n - 1} = \frac{\frac{1}{2} \left[ \frac{(k - 1)k}{2} + \frac{(2n - k)(2n - k + 1)}{2} \right]}{2n - 1} = \frac{k^2 - (2n + 1)k + n(2n + 1)}{2n - 1}.$$ 

Thus, the average value of the sum is

$$n \left( \frac{1}{2n} \sum_{k=1}^{2n} \frac{k^2 - (2n + 1)k + n(2n + 1)}{2n - 1} \right) = \frac{n(2n + 1)}{3}.$$ 

For $n = 10$ this gives an average of 70.