1. [Burris-Sanka. II.3.1] Let $\mathbf{A}$ be a $\Sigma$-algebra and $X \subseteq A$. Define a infinite sequence $E_0(X) \subseteq E_1(X) \subseteq E_2(X) \subseteq \cdots \subseteq A$ be recursion as follows. $E_0(X) = X$ and $E_{n+1}(X) = E_n(X) \cup \{ \sigma^A(a_1, \ldots, a_m) : m \in \omega, \sigma \in \Sigma_m, a_1, \ldots, a_m \in E_n(X) \}$. Prove that $\text{Sg}^A(X) = \bigcup_{n \in \omega} E_n(X)$.

2. Let $\Sigma$ be the signature of groupoid, i.e., a single binary operation. Consider the binary relations of subalgebra ($\subseteq$) and homomorphic image ($\preceq$) on the class of all $\Sigma$-algebras $\mathbb{A}(\Sigma)$. Prove that the $\subseteq ; \preceq = \preceq ; \subseteq$, i.e., prove that for all $A, B \in \mathbb{A}(\Sigma)$, if there exists a $C \in \mathbb{A}(\Sigma)$ such that $A \subseteq C \succ B$, then there exists a $D \in \mathbb{A}(\Sigma)$ such that $A \succ D \subseteq B$, and vice versa.

[Hint: The “vice versa” part is the harder to prove. Under the assumptions that $D \subseteq B$ and there exists an epimorphism $h : A \rightarrow D$, you have to construct a “superalgebra” $C$ of $A$ (i.e., $A \subseteq C$) and an epimorphism $g : C \rightarrow B$. It is helpful to draw pictures.

For simplicity you can assume that in this case $\Sigma$ is a groupoid signature, i.e., a single binary operation (written in infix notation). Without loss of generality we assume that $A$ and $B$ are disjoint (otherwise $B$ may first be replaced with an isomorphic image $B'$ and then at the end the epimorphism $g : C \rightarrow B'$ can be composed with the isomorphism from $B'$ to $B$). Let $C = A \cup (B \setminus D)$. Define $g : C \rightarrow B$ so that $g(c) = h(c)$ if $c \in A$ and $g(c) = c$ if $c \in B \setminus D$. Then define the operation $\cdot^C$ on $C$ so that it agrees with $\cdot^A$ on $A$ and the map $g$ is a homomorphism from $C$ to $B$. The definition of $c \cdot^C c'$ will require the consideration of several cases depending on whether or not $c$ and $c'$ are in $A$.]

3. Let $\mathbf{A}$ be a groupoid. Define a binary operation on the set $A^A$ of all mappings of $A$ into itself as follows. For all $f, g \in A^A$, $f \cdot g$ is the mapping from $A$ to itself such that, for all $a \in A$, $(f \cdot g)(a) = f(a) \cdot^A g(a)$. Prove that the set $\text{End}(A)$ of endomorphisms of $A$ is closed under this operation iff $\mathbf{A}$ satisfies the following entropic law. $(x \cdot y) \cdot (z \cdot w) \approx (x \cdot z) \cdot (y \cdot w)$. Prove that if $\mathbf{A}$ has an identity element (i.e., an element $e$ such that $e \cdot^A a = a = a \cdot^A e$ for all $a \in A$), then $\text{End}(A)$ is closed under $\cdot$ iff $\mathbf{A}$ is a commutative semigroup, i.e., satisfies the commutative law $x \cdot y \approx y \cdot x$ and the associative law $(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$.

4. A semigroup with identity $\langle A, \cdot, e \rangle$ is called a monoid. Prove that every cyclic monoid is commutative. Prove that $\mathbf{H}(\omega, +, 0)$, the class of all homomorphic images of $\langle \omega, +, 0 \rangle$, is the class of all cyclic (commutative) monoids. Use this result and the First Isomorphism Theorem to obtain a characterization of $\text{Co}(\omega, +, 0)$.

[hint: Show that for every monoid $A = \langle A, \cdot, e \rangle$ and every $a \in A$, there exists a (unique) homomorphism $h: \langle w, +, 0 \rangle \rightarrow A$ such that $h(1) = e$.]

(over)
5. A *Boolean algebra* is an algebra $B = \langle B, \lor, \land, -, 0, 1 \rangle$ such that $\langle B, \lor, \land, 0, 1 \rangle$ is a bounded distributive lattice and $-$ is the *complement* operation, i.e., $B$ satisfies the identities $-x \lor x \approx 1$ and $-x \land x \approx 0$.

(a) Prove that the complement of an element is unique, i.e., if $b \lor a = 1$ and $b \lor a = 0$, then $b = -a$. Prove the law of double negation $- - x \approx x$, and the two *DeMorgan laws*: $-(x \lor y) \approx -x \land -y$ and $-(x \land y) \approx -x \lor -y$.

(b) Let $I$ be an ideal of $B$ (in the sense of Problem #4 of Problem Set 1), and define a binary relation $\alpha$ on $B$ by $a \alpha b$ if $a - b, b - a \in I$, equivalently (since $I$ is an ideal), if $(a - b) \lor (b - a) \in I$. Prove that $\alpha$ is a congruence of $B$ and that $0/\alpha = I$.

(c) Let $\alpha$ be any congruence $B$. Prove that $0/\alpha = \{ b \in B : a \alpha a \}$ is an ideal of $B$, and that, for all $a, b \in B$, $a \alpha b$ if $a - b, b - a \in 0/\alpha$.

Thus there is a bijection between the ideals and congruences of $B$ that clearly preserves $\subseteq$. 