1. (10 points) a) Let $a, b, c \in \mathbb{R}^n$ with $|a_i| \geq \sum_{j=1, j \neq i}^n |a_j|$, $a_i \neq 0$, $|b_k| \geq \sum_{j=1, j \neq k}^n |b_j|$ and $i \neq k$. Suppose the vector $c$ is defined by $c_j := b_j - \frac{b_k}{a_i}a_j$, $1 \leq j \leq n$. Show that $|c_k| \geq \sum_{j=1, j \neq k}^n |c_j|$. 

b) Prove that if $|a_{jj}| \geq \sum_{i=1, i \neq j}^n |a_{ji}|$, then Gauss elimination without pivoting can be used to compute a decomposition of the form $A = L \cdot U$. 

2. (10 points) Suppose we have the iterative method $x_{k+1} = M x_k + c$ for finding the unique solution $x^*$ to $Ax = b$ where $A, M$ are $n \times n$ matrices and $x, b, c \in \mathbb{R}^n$. 

a) State a necessary and sufficient condition for the iterates $\{x_k\}$ to converge to $x^*$ for any starting point $x_0$. 

b) Give the choice of $M$ and $c$ for Jacobi method and Gauss-Seidel method, respectively. Construct a matrix $A$ for which the Jacobi method converges, but the Gauss-Seidel method does not. 

c) The SR iteration is obtained by taking the weighted average of the last iterate and the Jacobi iterate. Suppose Jacobi method converges, prove that the SR method with $0 < \omega \leq 1$ also converges, where $\omega$ is the parameter used in the SR iteration. 

3. (10 points) Let $f : D \to \mathbb{R}$ be a twice continuously differentiable function, for an open interval $D$. Suppose $x^*$ is the unique single root of $f(x) = 0$. Consider the Newton iterations $x_{k+1} = x_k - f'(x_k)^{-1}f(x_k)$, $x_0 \in D$, $k = 0, 1, 2, \ldots$. 

a) Find a small $\eta > 0$ so that if $|x_0 - x^*| < \eta$, then $|x_{k+1} - x^*| \leq |x_k - x^*|/2$, $k = 0, 1, 2, \ldots$. 

b) Prove that Newton’s iterations converge and converge quadratically, i.e., $|x_{k+1} - x^*| \leq C |x_k - x^*|^2$ for some constant $C > 0$. 

c) If $x^*$ is the $l$-th ($l > 1$) order root of $f = 0$, formulate a modified Newton’s iteration so that quadratic convergence can still be obtained. 

4. (10 points) Let $s(t) = s_i(t)$, $t \in [t_i, t_{i+1}]$, $i = 0, \ldots, n-1$ be a cubic spline interpolation of the points $\{(t_i, y_i) : i = 0, \ldots, n\}$ with $s''(t_0) = \ldots$
2

\( s''(t_n) = 0 \). Show that

\[
\int_{t_0}^{t_n} [s''(t)]^2 dt \leq \int_{t_0}^{t_n} [f''(t)]^2 dt
\]

for any twice continuously differentiable function \( f \) such that \( y_i = f(t_i), i = 0, \ldots, n \).

5 (10 points) Let \( a \leq x_0 < x_1 < \cdots < x_n \leq b \) be an arbitrary partition of the interval \([a, b]\). Show that there exist unique numbers \( \gamma_0, \gamma_1, \ldots, \gamma_n \) such that

\[
\sum_{i=0}^{n} \gamma_i P(x_i) = \int_{a}^{b} P(x) \, dx
\]

for all polynomials \( P \) with degree \( (P) \leq n \).

6. (10 points) Consider the family of semi-implicit Runge-Kutta methods

\[
k_1 = f(y_n + \beta hk_1), \quad k_2 = f(y_n + hk_1 + \beta hk_2),
\]

\[
y_{n+1} = y_n + h[(\frac{1}{2} + \beta)k_1 + (\frac{1}{2} - \beta)k_2].
\]

a) Apply this method to the problem \( y' = Ay \), where \( A \) is an \((N \times N)\) constant matrix. Obtain an expression of the form \( y_{n+1} = R(hA, \beta) y_n \), where \( R(hA, \beta) \) is a rational function of the matrix \( hA \). By comparing \( y_{n+1} \) to the exact solution \( y(x_{n+1}) = \exp(hA)y(x_n) \), under the localizing assumption \( y_n = y(x_n) \), determine the order and the principal part of the local truncation error.

b) Show that if \( \beta > \frac{1}{2} \) then the negative real axis \( \{z : \text{Im}(z) = 0, \text{Re}(z) < 0\} \), is contained in the region of absolute stability of the method.