1. (10 points) Given an \( n \times n \) real matrix \( A \), the following algorithm computes the factorization \( A = LU \), assuming it exists, where \( A(i,j) \) is overwritten by \( L(i,j) \) if \( i > j \) and by \( U(i,j) \) otherwise.

\[
\begin{align*}
\text{For } k = 1 \text{ to } n-1 \\
\text{For } i = k+1 \text{ to } n \\
A(i,k) = A(i,k)/A(k,k) \\
\text{End} \\
\text{For } j = k+1 \text{ to } n \\
\text{For } i = k+1 \text{ to } n \\
A(i,j) = A(i,j) - A(i,k)A(k,j) \\
\text{End} \\
\text{End} \\
\text{End}
\end{align*}
\]

(a) Use this algorithm to show that if \( A \) has upper bandwidth \( q \) (\( A(i,j) = 0 \) whenever \( j > i + q \)) and lower bandwidth \( p \) (\( A(i,j) = 0 \) whenever \( i > j + p \)), then \( U \) has upper bandwidth \( q \) and \( L \) has lower bandwidth \( p \).

(b) Modify the algorithm to avoid the unnecessary computations for a banded matrix \( A \) with upper bandwidth \( q \) and lower bandwidth \( p \), and show that the computation can be reduced to order of \( 2nqp \) floating point operations for \( q, p < n \).

2. (10 points) Let \( A \) be an \( n \times n \) complex, diagonalizable matrix and \( z^{(0)} \) a unit \( n \)-dimensional complex vector. Let \( ?_1, \ldots, ?_n \) be the eigenvalues of \( A \) and \( x^{(1)}, x^{(2)}, \ldots, x^{(n)} \) corresponding eigenvectors. Assume that \( |?_1| > \ldots > |?_n| \) and \( z^{(0)} = a_1 x^{(1)} + \ldots + a_n x^{(n)} \) and \( a_1 \neq 0 \). Consider the following Power iteration:

\[
y^{(k)} = A z^{(k-1)}, \quad \mu^{(k)} = f(y^{(k)}) / f(z^{(k-1)}), \quad z^{(k)} = y^{(k)} / \| y^{(k)} \|, \quad k = 1, 2, \ldots
\]

where \( f \) is a linear function and \( f(x^{(1)}) \neq 0 \). Show that \( \mu^{(k)} \) converges to \( ?_1 \).

3. (10 points) Let \( A \) be a nonsingular matrix and \( \{X_k\} \) be a sequence of matrices satisfying

\[
X_{k+1} = X_k + X_k (I - AX_k), \quad k = 0, 1, 2, \ldots
\]

(a) Show that if \( \|I - AX_0\| < 1 \), \( \{X_k\} \) converges to \( A^{-1} \). Further, \( E_k = I - AX_k \) satisfies \( E_{k+1} = E_k E_k \).

(b) Show that \( \{X_k\} \) converges locally quadratically. In addition, if \( AX_0 = X_0 A \), \( AX_k = X_k A \) for all \( k \).

4. (10 points) Let \( P_n[a,b] \) denote the set of all polynomials of degree less than or equal to \( n \) on the interval \([a, b]\). Assume \( p_0, p_1, \ldots, p_n \) form an orthonormal basis for \( P_n \) with respect to the inner product \( \langle p, q \rangle = \int_a^b p(x)q(x)dx \).

(a) Prove that for a given continuous function \( f \) on \([a, b]\), the polynomial \( p(x) = \sum_{j=0}^n a_j p_j \), where \( a_j = \int_a^b f(x)p_j(x)dx \), is the best \( L^2[a,b] \) approximation to \( f \) on the space \( P_n \) (i.e., prove that \( \| f - p \|_{L^2[a,b]} \leq \| f - q \|_{L^2[a,b]} \) for all \( q \in P_n \)).
(b) On the space $P_1[0, p]$ find the best $L^2[0, p]$ approximation to the continuous function $\sin x$.

5. (10 points) Let $Q(x)$ be a polynomial of degree $n$ which satisfies

$$\int_a^b Q(x)s(x)dx = 0 \quad \forall s(x) \in P_{n-1}[a, b].$$

($P_{n-1}[a, b]$ denotes the set of all polynomials of degree $n-1$ on $[a, b]$.) Assume $Q(x)$ has $n$ distinct roots $w_0, w_1, \ldots, w_{n-1}$.

(a) Prove that there exist $n$ constants $c_0, c_1, \ldots, c_{n-1}$ such that the integration rule

$$\int_a^b f(x)dx \approx \sum_{j=0}^{n-1} c_j f(w_j)$$

is exact for polynomials of degree $2n-1$ (Hint: First prove that there exist such constants such that

$$\int_a^b p(x)dx = \sum_{j=0}^{n-1} c_j p(w_j) \quad \forall p \in P_{n-1}[a, b]$$

and then $\forall p \in P_{2n-1}(a, b)$).

(b) Verify that polynomial $Q(x) = 3x^2 - 1$ satisfies

$$\int_{-1}^1 Q(x)l(x)dx = 0 \quad \forall l(x) \in P_1[-1, 1];$$

then find the distinct roots $w_0, w_1$ of $Q(x)$; finally, find constants $c_0$ and $c_1$ such that

$$\int_{-1}^1 p(x)dx = c_0 p(w_0) + c_1 p(w_1) \quad \forall p \in P_1[-1, 1].$$

6. (10 points) For the initial value problem, $y' = f(t, y)$, $y(0) = y_0$ on the interval $[0, T]$, consider an implicit two-step scheme

$$w_{n+1} = \frac{4}{3} w_n - \frac{1}{3} w_{n-1} + \frac{2h}{3} f(t_{n+1}, w_{n+1}),$$

$$w_0 = y_0, \quad w_1 = y_0 + hf(t_1, y_0),$$

where $h = T/N$ and $t_n = nh$.

(a) Check the consistency of the scheme. What is the order of the accuracy of the scheme?

(b) Check the stability of the scheme by analyzing the stability polynomial.

(c) Find the stability region for the scheme.

(d) If $f = -y$ and denote $e_n = y(t_n) - w_n$, prove that $(3+2h)e_{n+1} - 4e_n + e_{n-1} = O(h^2)$. Find the solution $e_n(h)$ at $h = 0$. Can you use this result to comment on the stability of the scheme?