Qualifying Examination in Analysis  
Spring 2001

- Write your student identification number on every page of the solutions you hand in. Do not write your name.
- Hand in a total of 6 problems, including at least 2 from each part. You will not get partial credit for attempting any more than 6 problems.
- To pass, you must get substantial credit from both Parts I and II.

**Part I. Real Analysis**

1. Let \((X, \mathcal{A}, \mu)\) be an arbitrary measure space with \(\mu\) a positive measure. Recall that a measure space is \(\sigma\)-finite if \(X\) can be written as a countable union of sets of finite measure. Prove that \((X, \mathcal{A}, \mu)\) is \(\sigma\)-finite if and only if there exists a strictly positive function \(f \in L^1(\mu)\).

2. Give an example of each of the following:
   a) A function \(f\) which is unbounded but Lebesgue integrable on \((0, \infty)\).
   b) A function \(f\) which is Lipschitz continuous but not differentiable everywhere.
   c) A function \(f\) which is absolutely continuous but not Lipschitz continuous on \([0, 1]\).
   d) A sequence \(\{f_n\}\) of continuous functions on \([0, 1]\) that converges pointwise to a function \(f\) on \([0, 1]\), but \(f\) is not continuous.
   e) A sequence \(\{f_n\}\) of functions that converges to zero pointwise on \([0, 1]\) but not in \(L^1([0, 1])\).
3. Let \( p, q > 1 \) satisfy \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \Omega \subset \mathbb{R}^N \).
   a) Show that if \( f_n \to f \) in \( L^p(\Omega) \) and \( g_n \to g \) in \( L^q(\Omega) \) then \( f_n g_n \to fg \) in \( L^1(\Omega) \).
   b) Explain carefully what is meant by the statement that \( L^q(\Omega) \) is the dual space of \( L^p(\Omega) \).

4. If \( f \in L^q(\mathbb{R}^N) \) for some \( q < \infty \), show that

\[
\lim_{p \to \infty} \|f\|_{L^p} = \|f\|_{L^\infty}
\]

Also, show by example that the conclusion may be false without the assumption that \( f \in L^q(\mathbb{R}^N) \).

5. Show that

\[
f(x) = \sum_{n=1}^\infty \frac{1}{n} \sin \left( \frac{x}{n + 1} \right)
\]

converges pointwise on \( \mathbb{R} \) and uniformly on each bounded interval of \( \mathbb{R} \) to a differentiable function \( f \) which satisfies \( |f(x)| \leq |x| \).

6. Prove the Riemann-Lebesgue Lemma: For any \( f \in L^1(\mathbb{R}) \)

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) e^{inx} \, dx = 0
\]

You may use the fact that for any \( f \in L^1(\mathbb{R}) \) and any \( \epsilon > 0 \) there exists a step function \( g \in L^1(\mathbb{R}) \) such that \( \int_{-\infty}^{\infty} |f(x) - g(x)| \, dx < \epsilon \).
Part II. Complex Analysis

1. Let $\Gamma = \{ z \in \mathbb{C} : |z - (6 + i)| = 3 \}$. Evaluate $\int_{\Gamma} (\bar{z} - i)^2 \, dz$ where the orientation of $\Gamma$ is in the counterclockwise direction.

2. Assume that $f, g$ are holomorphic in the disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$, continuous on $\overline{D}$ and have no zeros in $D$. If $|f(z)| \equiv |g(z)|$ for $|z| = 1$, prove that $f(z) = kg(z)$ in $D$, for some constant $k$ of modulus 1.

3. Evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$. (Suggestion: Integrate $\frac{\cot(\pi z)}{z^2 + 1}$ around a suitable contour.)

4. Prove that $F(z) = \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2}$ is defined and holomorphic for $z \neq \pm 1, \pm 2, \ldots$. 