ANALYSIS QUALIFYING EXAMINATION

Fall 1998
Saturday, August 15, 1998, 9:00am-12:00 noon
Room 408 Carver

Instructions:

a. **Write your social security number on every page that you turn in. Do NOT write your name on any sheet you turn in.**

b. Work no more than 6 problems, with no more than 4 from either Part I or Part II. No credit will be given for additional problems, and if additional problems are turned in, only the worst ones will be counted.

c. Work each problem on a separate sheet of paper and clearly indicate the part and problem number.

d. To pass you must receive substantial credit from each part. One correct problem will be counted more than two “half correct” problems in the grading.

**PART I. Complex Analysis**

1. Let $\gamma$ be the circle of radius 2 centered at the origin and with counterclockwise orientation. Determine the value of each of the following:
   
a. $\int_{\gamma} (z)^{n} \, dz$ (n an integer)  
b. $\int_{\gamma} \frac{e^{\sin z}}{e^{\pi z} - 2} \, dz$

2. Describe geometrically the domain of analyticity (with principle branch of the logarithm) of the function $f(z) = \log \frac{z - i}{1 - z}$.

3. Suppose $f(z)$ is analytic everywhere except at the origin, where it has a pole. Suppose $f(1/n) = n$ for $n = 1, 2, \ldots$
   
a. Show that $f(z) = z^{-1}$.
   
b. Show this is not necessarily true if it is only assumed that $f$ is analytic everywhere except at the origin. (Hint: First understand why $f$ is not uniquely determined.)

4. Determine the supremum of $|f(1/2)|$ over the class of all functions $f$ which are analytic in the unit disc and satisfy $|f(z)| \leq 4|z|^{1.3}$ in the unit disc. Find all functions $f$ (if any) for which the supremum is attained.
PART II. Real Analysis

In all the problems below, \( \mathbb{R} \) denotes the real line, \( \lambda \) denotes Lebesgue measure on \( \mathbb{R} \) and \( \mathcal{L} \) denotes the Lebesgue measurable sets on \( \mathbb{R} \).

1. Let \((X, \mathcal{M}, \mu)\) be a measure space and \( f : X \to \mathbb{R} \). Suppose that \( \{x : f(x) > \tau\} \) is measurable for each rational number \( \tau \). Is \( f \) measurable?

2. a. Give an example of a sequence \( \{f_n\}_{n=1}^{\infty} \) of nonnegative functions on the interval \( 0 \leq x \leq 1 \) that satisfies the following properties:
   (i) \( f_n \) is continuous, for \( n = 1, 2, 3, \ldots \).
   (ii) \( \{f_n(x)\}_{n=1}^{\infty} \) is unbounded for each \( x \) in \([0, 1]\).
   (iii) \( \lim_{n \to \infty} \int_0^1 f_n(x) dx = 0 \).

   b. Can a sequence satisfying (i)-(iii) also satisfy (iv) below?
   (iv) \( \liminf_{n \to \infty} f_n(x) > 0 \), for all \( x \in E \), for some \( E \subset [0, 1] \) with \( \lambda(E) > 0 \).

3. Let \( p \) and \( q \) be positive real numbers with \( \frac{1}{p} + \frac{1}{q} = 1 \), and let \( g \in L_q(= L_q(\mathbb{R}, \mathcal{L}, \lambda)) \).

   For \( f \in L_p \) and real \( y \) define the function \( T_y f \) by \( (T_y f)(x) = f(x-y) \), and the function \( Lf \) by
   \[
   (Lf)(y) = \int (T_y f)(x) g(x) \, d\lambda(x).
   \]

   Show that \( L \) is a continuous linear operator from from \( L_p \) to \( L_\infty \).

4. Let \((X, \rho)\) and \((Y, \sigma)\) be metric spaces and let \( f : X \to Y \) be onto and continuous.
   a. If \( X \) is separable is \( Y \) separable?
   b. If \( X \) is complete is \( Y \) complete?

5. Let \( S \) be a linear subspace of a Hilbert space \( H \) with the property that the only element of \( H \) orthogonal to every element of \( S \) is the zero element. Prove that \( S \) is dense in \( H \).

6. Let \( P \) be a polynomial and let \( f \) be a Lebesgue integrable real valued function defined on \( \mathbb{R} \). The function \( g \) is defined for real \( t \) by
   \[
   g(t) = \int P(\sin xt) f(x) \, d\lambda(x).
   \]

   Prove that \( g \) is continuous.