ANALYSIS QUALIFYING EXAMINATION

Fall, 2003
Saturday, August 23, 2003 9:00-12:00
Room 268 Carver

Instructions:

- Write your social security number on every page you hand in. Do NOT write your name on any sheet you turn in.
- Work no more than 6 problems. No credit will be given for more than 6 problems so if you turn in more than 6 problems, all problems will be graded and your total score will be based on your 6 worst problem scores.
- Work each problem on a separate piece of paper and clearly indicate the part and the problem number of that part.
- To pass you must work at least two problems from Part I and three problems from Part II. One correct problem will be counted more than two “half correct” problems in the grading.

PART I. Complex Analysis

1. Let $f$ be an integrable, real valued function defined on $[0, 1]$. For $z \in \mathbb{C} - [0, 1]$, where $\mathbb{C}$ is the set of complex numbers, define

$$F(z) = \int_0^1 \frac{f(x)}{x - z} \, dx.$$  

i. Show that $F$ is well defined on $\mathbb{C} - [0, 1]$.
ii. Show that $F$ is analytic on $\mathbb{C} - [0, 1]$.

2. Find the Laurenst expansion of

$$\frac{1}{z - 2} + \frac{1}{z - 3}$$

in the annulus $\{z \in \mathbb{C} : 2 < |z| < 3\}$.

3. Suppose that $f(\cdot)$ is analytic in $\{z \mid |z| < R\}$. Suppose that for every $a$ with $|a| < R$, at least one coefficient, $c_n$, of the power series expansion of $f(\cdot)$ about $a$,

$$f(z) = \sum_{n=0}^{\infty} c_n(z - a)^n$$

is zero. Prove that $f(\cdot)$ is a polynomial.

4. Evaluate, using contour integration:

$$\int_0^{\infty} \frac{\sin(\pi x)}{x(1 - x^2)} \, dx.$$  

5. Find all real $a, b$ such that $a + ib = i^{(i)}$. 

1
PART II. Real Analysis

1. Let $m$ be Lebesgue measure on $\mathbb{R}$, the real numbers. Three other measures on $\mathbb{R}$ are defined by

$$
\delta(E) = \begin{cases} 
1 & \text{if } 0 \in E \\
0 & \text{otherwise,}
\end{cases}
$$

$$
\mu(E) = m(E) + \delta(E),
$$

$$
\nu(E) = \int_E x \, dm + \delta(E).
$$

a. Show that $\nu$ is not absolutely continuous with respect to $m$

b. Show that $\nu$ is absolutely continuous with respect to $\mu$.

c. Compute the Radon-Nikodym derivative of $\nu$ with respect to $\mu$.

2. Let $p > 1$ and let $f_n : [0, 1] \to \mathbb{R}$, for $n = 1, 2, \ldots$ be a sequence of $L^p(m)$ functions where $m$ denotes Lebesgue measure. Suppose that $\sup\{|f_n|_p, n \geq 1\}$ is finite and $f_n \to f$ a.e. Show that $f \in L^p$ and that for $1 \leq \alpha < p$, $f_n \to f$ in $L^\alpha$.

3. Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers which is bounded above by some real number $M$. Some authors define $\limsup_{n \to +\infty} a_n$ to be $\sup\{b | b = \lim_{k \to +\infty} a_{n_k} \text{ where } \{a_{n_k}\}_{k=1}^\infty \text{ is a convergent subsequence of } \{a_n\}_{n=1}^\infty \}$

while others define it to be $\lim_{n \to -\infty} b_n$ where we set $b_n = \sup\{a_k \mid k = n, n+1, \ldots, \}$ for convenience. Carefully prove that these definitions are equivalent.

4. Let $f_n \to f$ a.e. where $f \in L^1(\mathbb{R})$. Show that $\int_{\mathbb{R}} |f_n(x) - f(x)| \, dx \to 0$ as $n \to \infty$ if and only if $\int_{\mathbb{R}} |f_n(x)| \, dx \to \int_{\mathbb{R}} |f(x)| \, dx$ as $n \to \infty$.

5. Assume $\{f_n\}$ is a sequence of measurable real-valued functions, and

$$
E = \{x : \lim_{n \to +\infty} f_n(x) \text{ exists as a real number}\}.
$$

Show that $E$ is measurable.

6. Let $f(x) = \sin(x^\alpha)$ on $(0, \infty)$ where $\alpha \in \mathbb{R}$, the real numbers. For what values of $\alpha$ is $f \in L^1(0, \infty)$? For what values of $\alpha$ does $\int_0^\infty f(x) \, dx$ converge in the sense of a proper or improper Riemann Integral? Support your answers with proofs.