ANALYSIS QUALIFYING EXAM

Fall 2002
August 24, 2002, 9:00 am - 12:00 noon
Room 408 Carver

Instructions

• Write your complete social security number on every page that you turn in. Do NOT write your name on any sheet that you turn in.
• Work 6 problems, with at least 2 from Part I and at least 3 from Part II. No credit will be given for additional problems, and if additional problems are turned in, only the worst ones will be counted.
• Work each problem on a separate sheet of paper, and clearly indicate the part and problem number.
• To pass, you must receive substantial credit from each part. One correct problem will be counted more than two “half correct” problems in the grading.

Part I: Complex Analysis

1. Evaluate \( \int_0^\infty \frac{x^2 \cos x}{1 + x^4} \, dx \).

2. Find the number of zeros of \( f(z) = z^6 + 5z^2 + z - 2 \) in the following regions:
   (a) \( \{z : 1 < |z| < 2\} \)
   (b) \( \{x + iy : x > 0\} \).

3. Suppose \( a_j \) are real numbers, with \( a_0 \geq a_1 \geq \cdots a_n \geq \cdots \) and \( \lim_{n \to \infty} a_n = 0 \). Show that \( \sum_{n=0}^\infty a_n z^n \) converges for all \( z \) with \( |z| = 1 \) and \( z \neq 1 \).

4. For each of the following cases, give an example of a function \( f \) holomorphic in \( D(0, 1) \) that satisfies the condition, or prove that such a function does not exist.
   (a) \( f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^2} \) for \( n = 2, 3, 4 \cdots \)
   (b) \( f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right) = \frac{1}{n^3} \) for \( n = 2, 3, 4 \cdots \)
Part II: Real Analysis

1. Let $s(x)$ be the “sawtooth” function defined on $[0, 2)$ by

$$s(x) = \begin{cases} x & \text{if } x \in [0, 1) \\ 2 - x & \text{if } x \in [1, 2) \end{cases}$$

and extended periodically to all of $\mathbb{R}$ (see sketch).

Let

$$f(x) = \sum_{k=0}^{\infty} 2^{-k} s(2^k x).$$

This function is continuous, but not differentiable anywhere in the classical sense (you don’t have to prove that). Show that this function has a derivative in the sense that there exists a locally integrable function $g$ so that

$$f(x) = \int_0^x g(t) \, dt.$$  

2. Let $\{f_n\}$ be a sequence of non-negative measurable functions that converge to $f$, and suppose $f_n \leq f$ for each $n$. Then

$$\int f = \lim_{n \to \infty} \int f_n.$$  

3. Prove that an uncountable set of points in the plane must have a limit point.

4. Show that if $f$ is measurable, so is $|f|$. Is the converse true? Why or why not? (Hint: Assume the existence of a nonmeasurable set.)

5. A set $E \subset [0, 1]$ has the property that there exists a constant $d > 0$ such that for any subinterval $(\alpha, \beta)$ of $[0, 1]$

$$m(E \cap (\alpha, \beta)) > d(\beta - \alpha).$$

Show that $m(E) = 1$. (Note: $m$ is Lebesgue measure).

6. Let $f \in L^1(\mathbb{R})$ and set, for $x \in \mathbb{R}$,

$$g(x) = \int_{\mathbb{R}} e^{ixy} f(y) \, dy.$$  

Prove that $g$ is uniformly continuous.