APPLIED MATH QUALIFYING EXAMINATION

Spring 2002
Saturday, Jan. 25 9:00am-12:00 noon
Room 408 Carver

Instructions:

• Write your social security number on every page that you turn in. Do NOT write your name on any sheet you turn in.
• Turn in solutions to 6 problems. No credit will be given for additional problems.
• Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. SHOW ALL WORK

(1) Find the Green’s function that satisfies
\[ \frac{d^2 g}{dx^2} - g = \delta(x - y), \quad -\infty < x, y < \infty, \quad \lim_{x \to \pm\infty} g(x, y) = 0. \]

Use g and the reflection principle to find a solution formula for
\[ u'' - u = f(x), \quad 0 < x < \infty, \quad \lim_{x \to \infty} u(x) = 0, \quad u(0) = 0. \]

(2) Let \( S = \{(x, t) : |x + t| \leq 1\} \) and let \( u(x, t) \) denote the characteristic function for \( S \), i.e., \( u(x, t) = 1 \) if \( (x, t) \in S \) and is zero otherwise. Calculate
\[ \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} \]

in the sense of distributions on \( \mathbb{R}^2 \).

(3) Let \( f \) be a locally integrable function on \( \mathbb{R} \). Show that \( \frac{f(x + h) - f(x)}{h} \) converges as \( h \to 0 \) to \( f' \) in the sense of distributions.

(4) Calculate the Fourier transform (in the sense of distributions if necessary) of \( f(x) = e^{-|x|} \) and \( g(x) = \cos x \). Use this to calculate \( f \ast g \), the convolution of \( f \) and \( g \). (Here, \( (f \ast g)(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\,dy \).)
(5) Let \( M = \{ w \in C^2([0, 1]) : w(0) = 0 \} \) and consider the following boundary value problem in weak form: Find \( u \in M \) such that

\[
\int_0^1 (u'v' + vw - h(x)v) \, dx + 2u(1)v(1) = 0, \quad \forall v \in M.
\]

Write down the strong form of this boundary value problem and prove that any solution \( u \in M \) of the weak BVP must also be a solution of the strong BVP. (Assume \( h \) is continuous.)

(6) Obtain a series solution for

\[
\begin{align*}
\frac{\partial u}{\partial t} - \pi \frac{\partial^2 u}{\partial x^2} &= \int_0^{2\pi} \cos(x - y)u(y, t) \, dy, \quad |x| < \pi, \ t > 0, \\
u(-\pi, t) &= u(\pi, t), \quad \frac{\partial u}{\partial x}(-\pi, t) = \frac{\partial u}{\partial x}(\pi, t) \quad t > 0, \\
u(x, 0) &= |x| \quad |x| < \pi.
\end{align*}
\]

Also determine the limiting steady-state solution. (Hint: To get started, write \( u(x, t) = a_0(t)/2 + \sum_{k=1}^{\infty} a_k(t) \cos kx + b_k(t) \sin kx \).)

(7) Let \( T \) be a bounded self-adjoint operator on a Hilbert space \( H \) and suppose that \( \langle Tx, x \rangle = 0 \) for all \( x \in H \). Show that \( T \) must be the zero operator. Find an example to show that this conclusion is false if \( T \) is not assumed to be self-adjoint.

(8) Let \( X \) be a Banach space, \( A \) a bounded operator on \( X \), \( \Lambda \) a bounded linear functional on \( X \), and \( x_0 \) a fixed element of \( X \). Define an operator \( B \) on \( X \) by \( Bx = Ax - \Lambda(x) x_0 \). Show that if \( A \) has a bounded inverse and \( \Lambda(A^{-1}x_0) \neq 1 \) then \( B \) has a bounded inverse.

(9) On \( L^2(0, 1) \) define the operator \( K \) by

\[
Ku = \int_0^1 k(x, y)u(y) \, dy
\]

where

\[
k(x, y) = \begin{cases} 
y(1-x), & 0 < y < x < 1 \\
x(1-y), & 0 < x < y < 1
\end{cases}
\]

Determine (precisely) \( \|K\| \). (Hint: \( k \) can be shown to be a Green’s function for some boundary value problem.)