1. Consider the integral equation

\[ u(x) + f(x) = \lambda \int_0^1 xyu(y) \, dy \quad 0 < x < 1 \]

a) For what values of \( \lambda \) is there one and only one solution \( u \) for a given \( f \in L^2(0,1) \)? For such \( \lambda \) find the solution.

b) For the remaining values of \( \lambda \), under what conditions on \( f \) does a solution exist? Find all solutions for such functions \( f \).

2. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary \( \partial \Omega \), and let \( f = f(x,y) \) be a given function defined on \( \partial \Omega \). Find the Euler-Lagrange equation for the problem of minimizing

\[ J(u) = \iint_{\Omega} \sqrt{1 + |\nabla u|^2} \, dydx \]

subject to the constraints that \( u = f \) on \( \partial \Omega \) and \( \iint_{\Omega} u^2 \, dydx = 1 \).
3. Consider a function \( v = v(x, t) \) defined implicitly by

\[
v(x, t) = f(x - tv(x, t)) \quad f \in C^1(R)
\]

a) Show that \( v \) satisfies \textit{Burger’s equation}, \( v_t + vv_x = 0 \), at any point \((x, t)\) for which \(1 + tf'(x - tv(x, t)) \neq 0\).

b) Let \( v \) be a solution as above, and \( S_\xi = \{(x, t) : x - tv(x, t) = \xi\} \) for a fixed constant \( \xi \). Show that \( v \) is constant on \( S_\xi \). Conclude that \( S_\xi \) is a straight line.

c) Suppose \( f'(\xi) < 0 \) for some \( \xi \). Prove that \( v \) cannot be continuous in the upper half plane \( \{(x, t) : t > 0\} \). (Hint: Find \( \xi_1 < \xi < \xi_2 \) so that the lines \( S_{\xi_1} \) and \( S_{\xi_2} \) intersect.)

4. Let

\[
f(x) = \begin{cases} 
\log x & x > 0 \\
0 & x < 0 
\end{cases}
\]

Show that \( f \in D'(\mathbb{R}) \) and compute \( f' \) in the sense of distributions.

5. a) Evaluate the convolution \( \delta' \ast H \) where \( \delta \) is the Dirac delta function and \( H \) is the Heaviside function.

b) Prove or disprove: If \( f, g \in L^1(\mathbb{R}^n) \) and \( f \ast g \equiv 0 \), then either \( f \) or \( g \) is zero.

6. Compute the Green’s function for

\[
u'' + 4u = f \quad 0 < x < 1 \quad u(0) + u'(0) = 0 \quad u(1) = 0
\]
7. Let $T$ be a bounded linear operator on a Hilbert space $H$ and $S = (I + T^*T)$.

   a) Show that $S$ is one to one and $||S^{-1}|| \leq 1$.
   b) If $T$ is compact, what can be said about the spectrum of $S$? Be as precise as possible.

8. Solve the boundary value problem for Laplace’s equation in the annulus $A = \{x : 1 < |x| < 2\} \subset \mathbb{R}^2$:

\[
\Delta u = 0 \quad x \in A \\
u(x) = 0 \quad |x| = 2 \\
u(x) = \begin{cases} 
0 & |x| = 1, \quad x_2 < 0 \\
1 & |x| = 1, \quad x_2 > 0 
\end{cases}
\]

by separation of variables.

9. Let $u$ be a continuous function and satisfy $u_{tt} - u_{xx} = 0$ in $\mathcal{D}'(\Omega)$, where $\Omega$ is an open subset of the $(x, t)$ plane, and suppose that the closed rectangle with vertices $(x, t), (x, t+2h), (x\pm h, t+h)$ is contained in $\Omega$.

   a) Show that $u$ must satisfy the four point property

\[
u(x, t) + u(x, t + 2h) = u(x - h, t + h) + u(x + h, t + h)
\]

   b) Show that there can be no solution of the following Dirichlet problem in the circle $D = \{(x, t) : x^2 + t^2 < 1\}$:

\[
u_{tt} - u_{xx} = 0 \quad (x, t) \in D \quad u(x, t) = x^2 \quad (x, t) \in \partial D \quad u \in C(\bar{D})
\]