**APPLIED MATH QUALIFYING EXAMINATION**

Fall 2000
Saturday, August 26 9:00am-12:00 noon
Room 408 Carver

Instructions:

a. Write your social security number on every page that you turn in. Do NOT write your name on any sheet you turn in.

b. Turn in solutions to 6 problems. No credit will be given for additional problems.

c. Start each problem on a separate sheet of paper, with the problem number clearly stated at the top. SHOW ALL WORK

1. Find the Green's function \( g(x, \xi) \) that satisfies

\[
\frac{d}{dx} ((x + 1) \frac{dg}{dx}) = \delta(x - \xi) \quad 0 < x, \xi < 1; \quad g(0, \xi) = \frac{dg}{dx}(1, \xi) = 0.
\]

Use the Green's function to write down in terms of an integral representation, the solution to

\[
((x + 1)u')' = f(x) \quad 0 < x < 1; \quad u(0) = u'(1) = 0.
\]

2. Prove that \( \lim_{m \to \infty} m^p \sin mx \to 0 \) in the sense of distributions for any integer \( p \geq 0 \).

3. Let \( J(u) = \int_0^1 (au^2 + (u')^2) \, dx \), with \( a > 0 \). Find the function \( u \) that minimizes \( J \) over the class \( \mathcal{A} \) of continuously differentiable functions on \([0, 1]\) that satisfy \( u(0) = 0, \, u(1) = 1 \).

4. Consider the operator \( A \) on \( L^2(-1,1) \) defined

\[
(Au)(x) = x \int_{-1}^{1} y^2 u(y) \, dy.
\]

Determine \( \mathcal{N}(A), \mathcal{N}(A^*), \mathcal{R}(A), \) and \( \mathcal{R}(A^*) \). What points are in the spectrum of \( A \)?

5. Let \( R \) denote the rectangle: \( 0 \leq x \leq a; \, 0 \leq y \leq \pi \). Let \( \mathcal{C} \) denote the class of differentiable functions on \( R \) that vanish on the boundary of \( R \). Determine the minimum value of \( J(u) \) over \( u \in \mathcal{C}, \, u \neq 0 \) where

\[
J(u) = \frac{\int_R (\frac{\partial u}{\partial x})^2 + (\frac{\partial u}{\partial y})^2 \, dx \, dy}{\int_R u^2 \, dx \, dy}.
\]

What is a function that minimizes \( J \)?
6. Let \( M = \{ f \in C[0,1] : 0 \leq f(x) \leq 1 \forall x \in [0,1] \} \). For \( f \in M \) define the operator \( T \) by

\[
(Tf)(x) = \frac{x}{3} + \alpha \int_0^1 xt(f(t))^2 \, dt. \quad (0 < \alpha < 1)
\]

i) Show that \( T \) is a contraction on \( M \). ii) Determine a fixed point for \( T \) in \( M \).

7. Find the solution of the Cauchy problem

\[
x u_x + -y u_y = 2u, \quad u(1, y) = f(y),
\]

where \( f \) is a continuously differentiable function on \((-\infty, \infty)\).

8. Let \( f \) be a distribution on \( \mathcal{R} \) that satisfies \( f' = 0 \) in the sense of distributions. Show that there is a constant \( C \) such that \( f = C \). That is

\[
\langle f, \phi \rangle = C \int_{\mathcal{R}} \phi \, dx, \quad \forall \phi \in C_0^\infty(\mathcal{R}).
\]

9. Let \( \{e_n\}_{n=1}^\infty \) be an orthonormal set in a Hilbert space \( H \). i) Show that \( \langle u, e_n \rangle \to 0 \) as \( n \to \infty \) for all \( u \in H \). ii) Show that if \( K \) is a compact linear operator in \( H \) then \( K e_n \to 0 \) as \( n \to \infty \).