1. A particle moves through three dimensional space with velocity
\[ \mathbf{v}(t) = \langle \sec^2 t, 2 \sec t \tan t, \tan^2 t \rangle. \]
At time \( t = 0 \) the particle is at \( \langle 0, 1, 2 \rangle \), find the position function of the particle for \(-\pi/4 \leq t \leq \pi/4\).

If \( \mathbf{r}(t) \) is the position function then \( \mathbf{r}'(t) = \mathbf{v}(t) \). So taking antiderivatives we have

\[
\mathbf{r}(t) = \int \mathbf{v}(t) \, dt = \left( \int \sec^2 t \, dt, \int 2 \sec t \tan t \, dt, \int \tan^2 t \, dt \right) \\
= \left( \tan t + C, 2 \sec t + D, \int (\sec^2 t - 1) \, dt \right) \\
= \left( \tan t + C, 2 \sec t + D, \tan t - t + E \right).
\]

Two of the three integrals are straightforward. The last one is the trickiest but this follows by relating \( \tan^2 t \) (something which we cannot directly integrate) to \( \sec^2 t \) (something which is easy to integrate). Now all that is left is to determine the constants \( C, D, E \). We have

\[
\mathbf{r}(0) = \langle C, 2 + D, E \rangle = \langle 0, 1, 2 \rangle
\]
giving the constants we need. So we have that the position function of the particle is

\[
\mathbf{r}(t) = \left( \tan t, 2 \sec t - 1, \tan t - t + 2 \right).
\]

(The condition for \(-\pi/4 \leq t \leq \pi/4\) is not needed directly, it only is used to guarantee that we stay away from the vertical asymptotes (solutions cannot be pushed past vertical asymptotes, something you will learn in your future math classes).)
2. Find the cumulative length function $s(t)$ (starting from $a = 1$) of the parametric curve $\langle \ln t, \sqrt{2}t, \frac{1}{2}t^2 \rangle$.

We have that the cumulative arc length function will be

$$s(t) = \int_a^t |\mathbf{r}'(u)| \, du.$$  

We are told that $a = 1$ and we now compute the derivative. We have

$$\mathbf{r}'(t) = \left\langle \frac{1}{t}, \sqrt{2}, t \right\rangle.$$  

Therefore

$$s(t) = \int_1^t \left| \left\langle \frac{1}{u}, \sqrt{2}, u \right\rangle \right| \, du = \sqrt{\left( \frac{1}{u} \right)^2 + (\sqrt{2})^2 + (u)^2} \, du = \int_1^t \sqrt{\frac{1}{u^2} + 2 + u^2} \, du = \int_1^t \sqrt{\left( \frac{1}{u} + u \right)^2} \, du = \int_1^t \left( \frac{1}{u} + u \right) \, du = \left( \ln u + \frac{1}{2}u^2 \right)|_1^t = \ln t + \frac{1}{2}t^2 - \frac{1}{2}.$$

(Frequently in this type of problem the functions will be chosen so that the terms on the inside of the square root “miraculously” combine into a perfect square. Of course this is because they have been rigged to do so and would not happen by coincidence.)
3. Sketch in the $xy$-plane the domain of $f(x, y) = \frac{\sqrt{4 - y^2}}{\ln(y - x^2)}$.

We look for our three potential problems, and in this case we have all sorts of problems.

- **Division by zero:** The denominator will be zero when the term inside of the log is 1, therefore our domain has the restriction that it is all $(x, y)$ so that $y - x^2 \neq 1$ or $y \neq x^2 + 1$.

- **Square root of a negative:** We need that the term inside the square root is non-negative, therefore our domain has the restriction that it is all $(x, y)$ so that $4 - y^2 \geq 0$, or $y^2 \leq 4$ which is equivalent to $-2 \leq y \leq 2$.

- **Log of a non-positive:** We need that the term inside the log is positive, therefore our domain has the restriction that it is all $(x, y)$ so that $y - x^2 > 0$, or $y > x^2$.

Putting all of these together we get the following picture.
4. Given the implicitly defined surface $z + \sin z = xy$ find $\partial^2 z / \partial x \partial y$ only in terms of $z$. (Hint: find both $\partial z / \partial x$ and $\partial z / \partial y$ then take the derivative of one of these with respect to the appropriate variable to find $\partial^2 z / \partial x \partial y$, at the end a substitution from the original relationship defining the surface then gives the desired form.)

Let us follow the hint. First we compute the two partial derivatives. Taking the derivative of both sides with respect to $x$ and with respect to $y$ we have the following (remember we think of $z = z(x, y)$):

$$\frac{\partial z}{\partial x} + (\cos z) \frac{\partial z}{\partial x} = y \quad \text{or} \quad \frac{\partial z}{\partial x} = \frac{y}{1 + \cos z},$$

$$\frac{\partial z}{\partial y} + (\cos z) \frac{\partial z}{\partial y} = x \quad \text{or} \quad \frac{\partial z}{\partial y} = \frac{x}{1 + \cos z}.$$

We now use the quotient rule to find

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{x}{1 + \cos z} \right) = \frac{(1 + \cos z) - x(-\sin z) \frac{\partial z}{\partial x}}{(1 + \cos z)^2}$$

$$= \frac{(1 + \cos z) + x \sin z \left( \frac{y}{1 + \cos z} \right)}{(1 + \cos z)^2} = \frac{(1 + \cos z)^2 + xy \sin z}{(1 + \cos z)^3}$$

$$= \frac{(1 + \cos z)^2 + (z + \sin z) \sin z}{(1 + \cos z)^3}.$$

At the last step we used the original relationship $z + \sin z = xy$ to replace the $xy$ term.

You can also simplify it further using the Pythagorean identity to get the following (though the form given above is perfectly fine):

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{2 + 2 \cos z + z \sin z}{(1 + \cos z)^3}.$$
5. Show that \( u(x, t) = \sin (x + \sin t) \) is a solution to the partial differential equation \( u_t u_{xx} = u_x u_{tx} \).

To see if it is a solution we need to compute the various partial derivatives and see if this function satisfies the equation. We have the following:

\[
\begin{align*}
    u_x &= \cos (x + \sin t) \\
    u_t &= \cos (x + \sin t) \cos t \\
    u_{xx} &= -\sin (x + \sin t) \\
    u_{tx} &= -\sin (x + \sin t) \cos t
\end{align*}
\]

Therefore we have

\[
\begin{align*}
    u_t u_{xx} &= (\cos (x + \sin t) \cos t)( -\sin (x + \sin t)) \\
    &= (\cos (x + \sin t))( -\sin (x + \sin t) \cos t) = u_x u_{tx}
\end{align*}
\]

showing that \( u \) is a solution to the differential equation.
6. Find the tangent plane to \( f(x, y) = x^3y - 3xy^2 \) at \((2, 1)\).

The equation for the tangent plane is given by

\[
z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b).
\]

We already know that \((a, b) = (2, 1)\) and that \(f(2, 1) = 2\) so we need the partial derivatives. We have

\[
\frac{\partial f}{\partial x}(x, y) = 3x^2y - 3y^2, \quad \text{and}
\]

\[
\frac{\partial f}{\partial y}(x, y) = x^3 - 6xy,
\]

so that

\[
\frac{\partial f}{\partial x}(2, 1) = 9, \quad \text{and}
\]

\[
\frac{\partial f}{\partial y}(2, 1) = -4.
\]

Substituting all of the values in we get our tangent plane

\[
z = 2 + 9(x - 2) - 4(y - 1) \quad \text{or} \quad z = 9x - 4y - 12
\]