This test is closed book and closed notes. No sophisticated calculator is allowed for this test. For full credit show all of your work (legibly!). If you know about L’Hospital’s method, then do not use it to answer any question; if you don’t know about L’Hospital’s method, then you should definitely not use it to answer any question! Near \( x = 3 \) a well chosen one is \( \frac{\sin(x-3)}{\sin(x-3)} \). Each problem is worth 10 points (a total of 50 points).

1. Find \( a \) and \( b \) so that \( f(x) = 2^{x+1} + ax^2 + bx + \pi^{137} \) satisfies the following:

- The average rate of change between \(-1\) and \(0\) is \(4\).
- The average rate of change between \(0\) and \(2\) is \(3\).

We need
\[
\frac{f(0) - f(-1)}{0 - (-1)} = 4 \quad \text{or} \quad f(0) - f(-1) = 4
\]
\[
\frac{f(2) - f(0)}{2 - 0} = 3 \quad \text{or} \quad f(2) - f(0) = 6
\]

\[
4 = f(0) - f(-1) = \left(2 + \pi^{137}\right) - \left(1 + a - b + \pi^{137}\right) = 1 - a + b
\]
\[
6 = f(2) - f(0) = \left(8 + 4a + 2b + \pi^{137}\right) - \left(2 + \pi^{137}\right) = 6 + 4a + 2b
\]

\[
\begin{align*}
-a + b &= 3 \\
4a + 2b &= 0
\end{align*}
\]

\[
\begin{align*}
-a + b &= 3 \quad \Rightarrow \quad b = a + 3 \\
2a + b &= 0 \quad \Rightarrow \quad 2a + (a + 3) = 0 \\
a &= -1 \\
b &= 2
\end{align*}
\]
2. Find \( \lim_{x \to \infty} \sin \left( \frac{2}{x} \right) \left( \sqrt{x^4 + 5x^3} - x^2 \right) \).

\[
\lim_{x \to \infty} \sin \left( \frac{2}{x} \right) \left( \sqrt{x^4 + 5x^3} - x^2 \right) \cdot \frac{\left( \sqrt{x^4 + 5x^3} + x^2 \right)}{\left( \sqrt{x^4 + 5x^3} + x^2 \right)}
\]

\[
= \lim_{x \to \infty} \sin \left( \frac{2}{x} \right) \frac{x^4 + 5x^3 - x^4}{\sqrt{x^4 + 5x^3} + x^2}
\]

\[
= \lim_{x \to \infty} \frac{\sin \left( \frac{2}{x} \right) \cdot 5x^3}{\sqrt{x^4 + 5x^3} + x^2} \cdot \frac{1}{x^2}
\]

\[
= \lim_{x \to \infty} \frac{5x \sin \left( \frac{2}{x} \right)}{\sqrt{1 + \frac{5}{x}}} \sim ?
\]

\[
= \frac{10}{2} = \boxed{5}
\]
3. Suppose that $f(x)$ is as shown in the figure below.

(a) List and classify all discontinuities for $0 \leq x \leq 4$ for $f(x)$.

- jump discontinuity at $x=1$
- removable discontinuity at $x=3$

(b) Find $\lim_{{x \to 1}} (f(x))^2$ or explain why the limit does not exist.

$\lim_{{x \to 1^-}} (f(x))^2 = 1 \cdot (-2)^2 = 4$

$\lim_{{x \to 1^+}} (f(x))^2 = (2)^2 = 4$

Match so we conclude

$\lim_{{x \to 1}} (f(x))^2 = 4$
4. Find \( \lim_{{x \to 3}} \frac{\sin((x-3))}{2x + \sqrt{5x+1} - 10} \).

Hint: multiplying by some well chosen ones would be helpful.

\[
\lim_{{x \to 3}} \frac{\sin((x-3))}{\sqrt{5x+1} - (10-2x)} \cdot \frac{\sqrt{5x+1} + (10-2x)}{\sqrt{5x+1} + (10-2x)}
\]

\[
= \lim_{{x \to 3}} \frac{\sin((x-3)) \cdot (\sqrt{5x+1} + (10-2x))}{(5x+1) - (10-2x)^2}
\]

\[
= \lim_{{x \to 3}} \frac{\sin((x-3)) \cdot (\sqrt{5x+1} + (10-2x))}{-(x-3)(4x-33)}
\]

\[
= \lim_{{x \to 3}} \frac{\sin((x-3))}{\sin(x-3)} \cdot \frac{\sin(x-3)}{(x-3)} \cdot \frac{\sqrt{5x+1} + 10-2x}{-(4x-33)}
\]

\[
\frac{8}{21}
\]
5. Using *any* method you like determine \( g'(0) \) where \( g(x) \) is defined piecewise as follows:

\[
g(x) = \begin{cases} 
x^2 \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0; \\
0 & \text{if } x = 0.
\end{cases}
\]

Make sure to explain your answer and what tools you used. (The function \( g(x) \) is continuous; you do not need to worry about continuity.)

Let us use the limit definition

\[
g'(0) = \lim_{h \to 0} \frac{g(0 + h) - g(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = \lim_{h \to 0} h \sin(\frac{1}{h})
\]

\( -1 \leq \sin(\frac{1}{h}) \leq 1 \)

\( -|h| \leq h \sin(\frac{1}{h}) \leq |h| \)

\( \text{does not converge, but bounded} \)

\( = 0 \) by Squeeze

**Fun note:** For those who know more about derivatives we have for \( x \neq 0 \) that by product/chain rule that

\[
g'(x) = 2x \sin \left( \frac{1}{x} \right) - \sin \left( \frac{1}{x} \right)
\]

and then \( \lim_{x \to 0} g'(x) \) does not exist!!

A very curious behavior. We must be careful going forward.